Approximation of continuous space systems and associated metrics and logics

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MLQA 2011

Mostly joint work with
Vincent Danos, Vineet Gupta, Radha Jagadeesan, Prakash Panangaden.

Desirable properties of approximants of $\mathcal{S}$?

A **(countable?) family** of finite-state systems that satisfy...

- below $\mathcal{S}$, or simulated by $\mathcal{S}$ (may do less)
- converge to $\mathcal{S}$

  e.g. could be approximated by

  - as a sequence, w.r.t. some distance $d(\mathcal{S}_i, \mathcal{S}) \to 0$
  - in properties: set of properties satisfied by $\mathcal{S}_i$ increases to set of properties satisfied by $\mathcal{S}$
- If $\mathcal{S}$ is finite, we can recover $\mathcal{S}$ itself?
- (freeness to guide approximation w.r.t. some constraints.)

Hence, we do not want to split the state space randomly.

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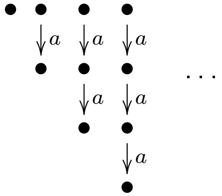
e.g.  could be approximated by 

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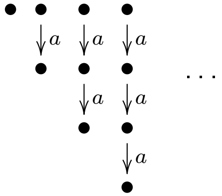
e.g.  could be approximated by 

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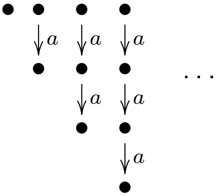
  e.g.  could be approximated by 

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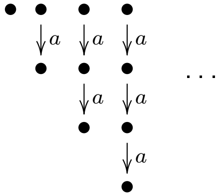
could be approximated by 

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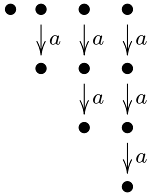
could be approximated by 

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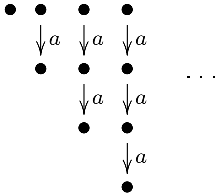
  could be approximated by 

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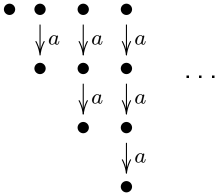
  could be approximated by 

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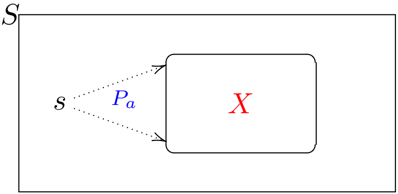
Outline

Labelled Markov processes

$(S, i, \Sigma, \{P_a\}_{a\in\mathcal{A}})$
S can be continuous



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$P_a(s, X)$: probability that the process in state s jumps to a state in X, with action a.

$P_a(\cdot, X) : S \to [0;1]$ is measurable
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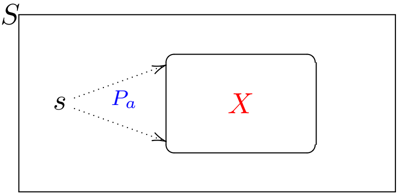

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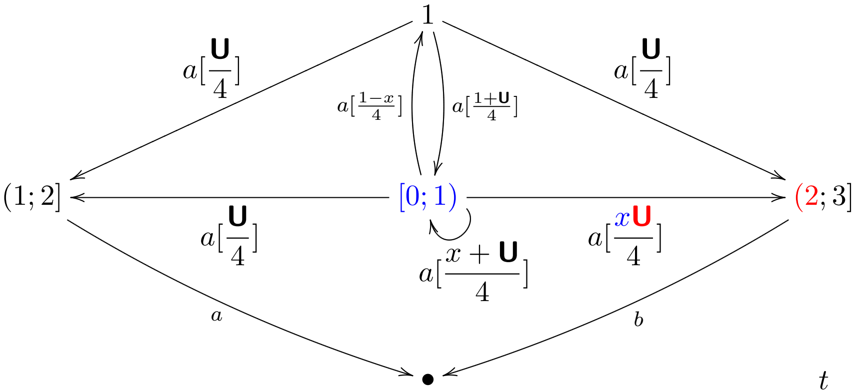
Example

U is the uniform distribution



For $x \in [0;1)$, $P_a(x,(2;2+y]) = \frac{xy}{4}$
Time is discrete but state space is continuous $\{\bullet\} \cup [0;3]$.

Notions of simulation, bisimulation on LMPs and a logic.

$$\mathcal{L}_\vee := \top \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \langle a \rangle_{\geq q}\phi. \qquad q \in \mathbb{Q} \cap [0;1]$$

Examples:

s_i is *simulated by* t_i

s_1 satisfies the formula $\langle a \rangle_{\geq 1/2}\langle b \rangle_{\geq 1}\top$

$t_1 \models \langle a \rangle_{\geq 1/3}\langle b \rangle_{\geq 1}\langle a \rangle^{n}_{\geq 1}\top$

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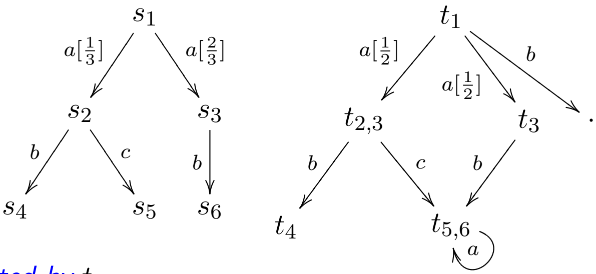

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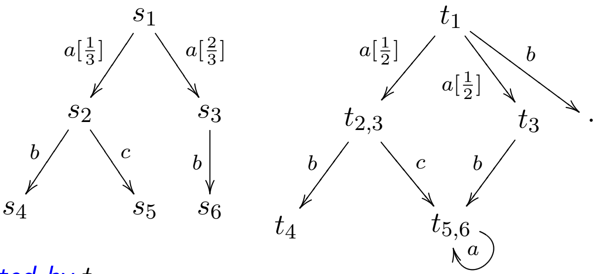

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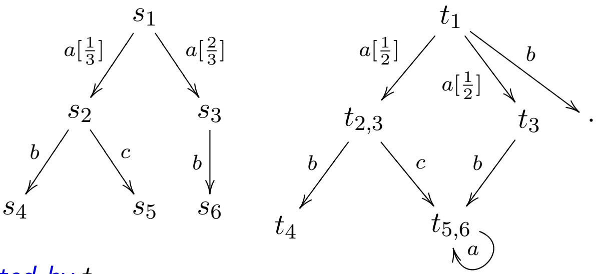

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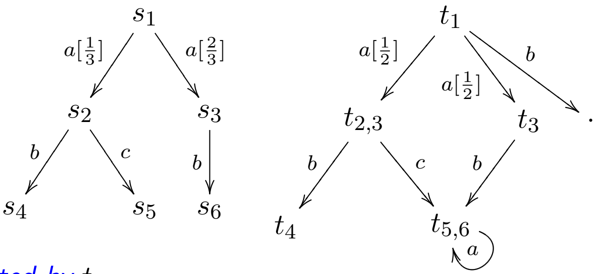

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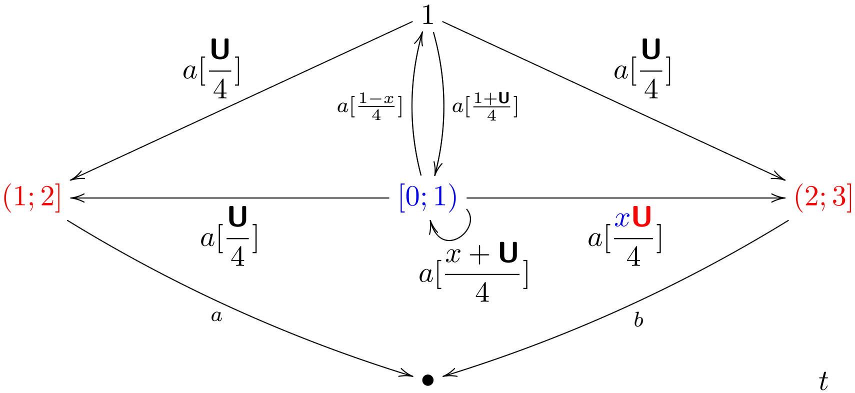
Bisimulation minimisation

States in $(1;2]$ are all bisimilar, similarly for $(2;3]$.



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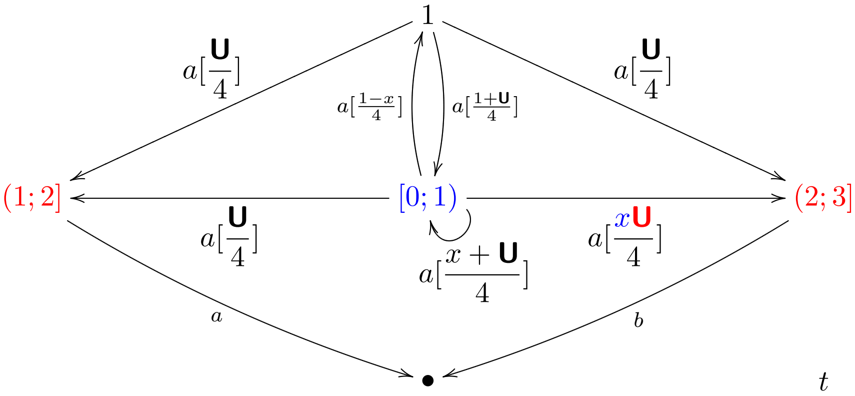

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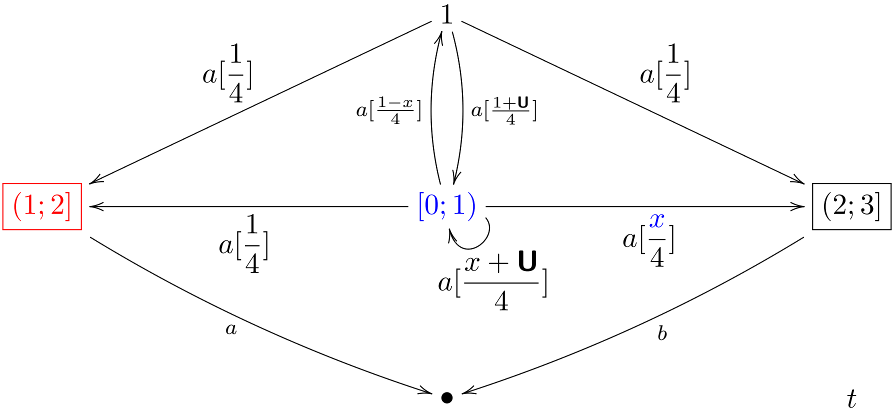

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State space is still continuous.

Outline

$\mathcal{S}(n,\epsilon)$ of depth n, precision ϵ [DGJP00] [DD03]

Approximants $\mathcal{S}(n,\epsilon)$ are defined according to depth n and precision ϵ.

State space is constructed by level,

Each level is a partition of the state space w.r.t. precision ϵ

Partition obtained from probabilities to previous level $P_a(\cdot, C_{l-1})$

Transitions are

- between states of the same level $P_a^{app}(X_l, C_l) := \inf_{x\in C_l} P_a(x, C_l)$
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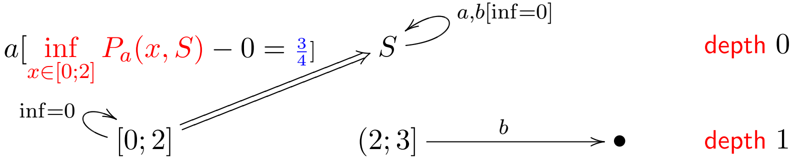
Example: part of $\mathcal{S}(2, 1/2)$.

At level 1, split S w.r.t. $P_a(\cdot, S)$ & $P_b(\cdot, S)$ values in $\{0\}, (0; \frac{1}{2}], (\frac{1}{2}; 1]$

At level 2, split S w.r.t. $\{0, \frac{1}{6}, \frac{2}{6}, \dots\}$



States in $[0;1]$ have probability $3/4$ of jumping to S.

Let us focus on state $(\frac{1}{3}; \frac{2}{3})$.

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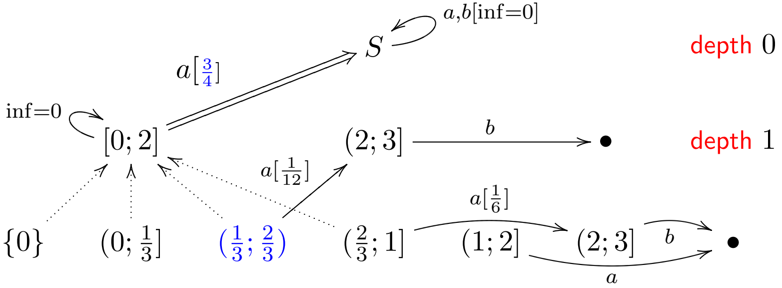

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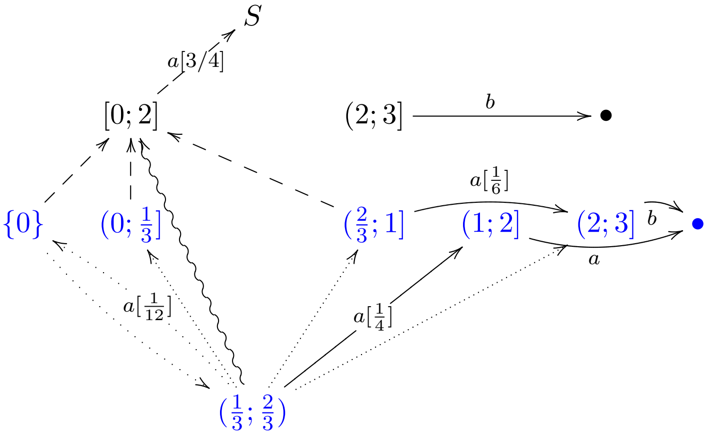

State $(\frac{1}{3};\frac{2}{3})$ has probability $3/4$ to $[0;2]$. It has $7/12$ probability to states of the same level that refine $[0;2]$. The remaining probability $2/12$ is sent to level 1.

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- ✓ converge to $\mathcal{S}$
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- ✓ If $\mathcal{S}$ is finite, it is its own approximant (for some ϵ and n)
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$\mathcal{S}(n, \epsilon)$ of depth n, precision ϵ

An approximation algorithm for labelled Markov processes: towards realistic approximation
Bouchard-Cote, Ferns, Panangaden, Precup, QEST '05.

Outline

Approximate through properties

Last scheme aggregates states that satisfy the same properties from some set.
For $\epsilon = 1/2$ and $n = 2$, the formulas are

- $\langle a_0 \rangle_{>q_0} \top$ for $a_0 \in \mathcal{A}, q_0 \in \{\frac{1}{2}, 1\}$ (depth 1)
- $\langle a_0 \rangle_{>q_0} (\wedge_i \langle a_i \rangle_{>q_i} \top)$ for $a_i \in \mathcal{A}, q_i \in \{\frac{1}{6}, \frac{2}{6}, \ldots, 1\}$ (depth 2)

The second scheme aims at doing the same but for any set of formulas

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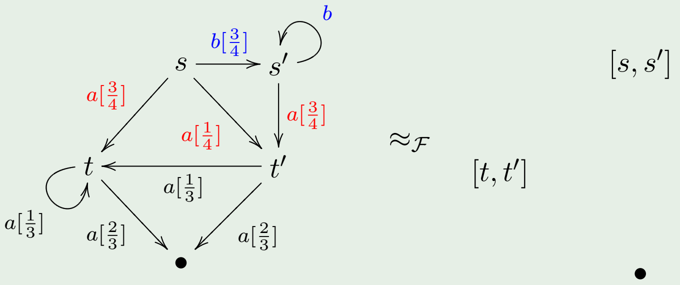
Approximate through properties

Quotient the state space w.r.t. a chosen set $\mathcal{F}$ of properties from some logic,

Example

Let $\mathcal{F} = \{\langle a\rangle_q \top,\ \langle b\rangle_q \top \mid q \in \{\frac{1}{2}\}\}$.



And transitions? Can we take infima?

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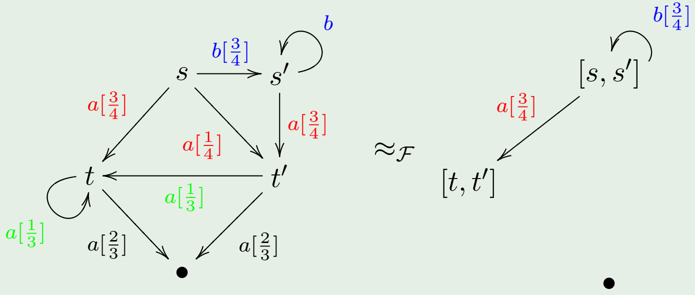

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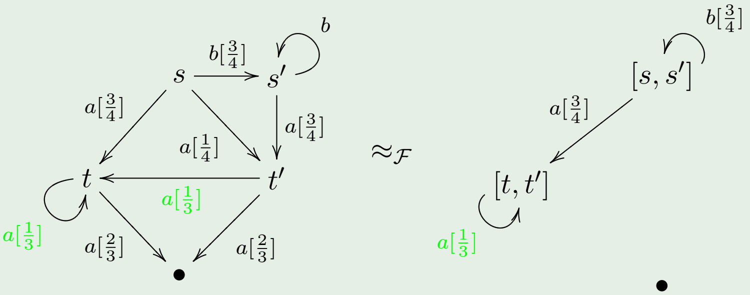

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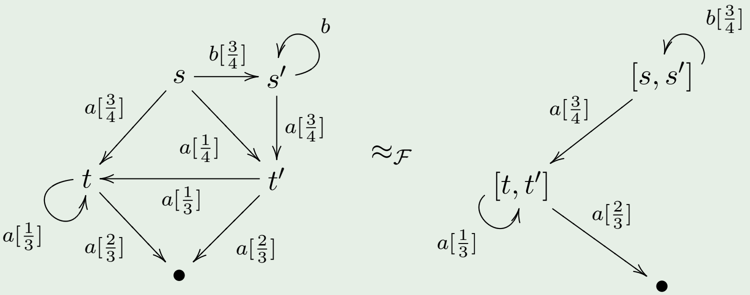

It works!

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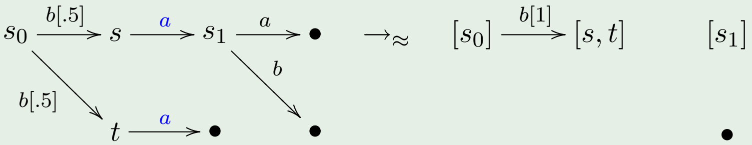

Approximate through properties

Quotient the state space w.r.t. a chosen set $\mathcal{F}$ of properties,
$P_a^{app}(C, D) = \inf_{x \in C} P_a(x, D)$

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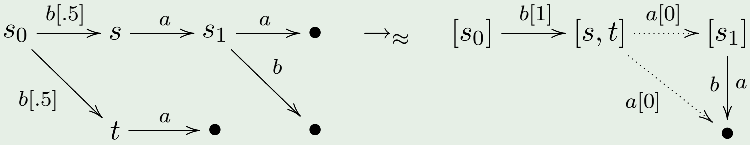

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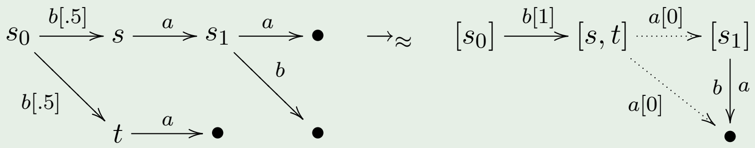

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A pre-LMP is a LMP where $P_a(s,-)$ satisfies

- $\forall A, B \in \Sigma$ disjoint

$$P_a(s, A \cup B) \quad \geq \quad P_a(s, A) + P_a(s, B)$$

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If R is an equivalence relation with measurable equivalence classes, the inf-quotient w.r.t R is a pre-LMP

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Approximate through properties

Theorem

Let $\mathcal{F} \subseteq \mathcal{L}^*$, $s \in S$. Then the quotient is a pre-LMP and

$$s \approx_{\mathcal{F}} [s]_{\mathcal{F}}$$

i.e.: the inf-quotient defines an $\approx_{\mathcal{F}}$-approximant

This is the best approximant below $\mathcal{S}$.
If $\mathcal{F}$ is finite, we get a finite approximant.
if $\mathcal{S}$ is finite, we get itself as an approximant when $\mathcal{F}$ is rich enough.

Desirable properties of approximants?

A **countable family** of finite-state systems that satisfy...

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Pre-LMPs have nice other properties (Concur 09)

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Maybe averaging could help us stay in the world of LMPs

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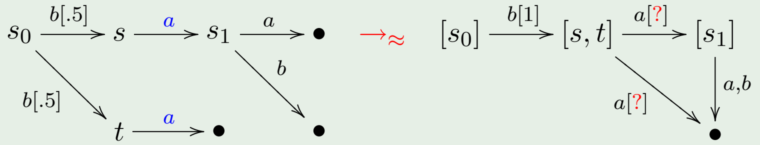

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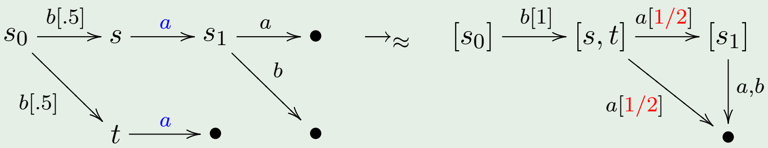

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Here we chose uniform: we treated s and t equally.

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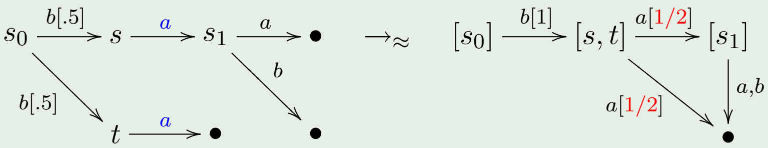

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where $\mathbb{E}_p$ is the conditional expectation (unique under condition)
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Outline

Metric defined as real valued logic

Definition

$\forall c \in (0,1]$, a family $\mathcal{F}^c$ of functional expressions generated by

$$f := 1 \mid 1 - f \mid \langle a \rangle f \mid \min(f_1, f_2) \mid f \ominus q \mid \quad q \in \mathbb{Q}$$

With the following semantics $f : \mathcal{S} \longrightarrow [0,1]$

$$\begin{aligned} \langle a \rangle f(s) &:= c \int_S f(t) P_a(s, dt), \\ f \ominus q(s) &:= \max(f(s) - q, 0), \end{aligned}$$

Definition

$$d^c(s,t) := \sup_{f \in \mathcal{F}^c} |f(s) - f(t)|$$

Papers on metric defined as real valued logic

- Metrics for labelled Markov processes,
  Desharnais, Gupta, Jagadeesan, Panangaden
  CONCUR '99 (and TCS 2004).
- The metric analogue of weak bisimulation for probabilistic processes,
  same authors, LICS '02.
- Approximating a behavioural pseudometric without discount,
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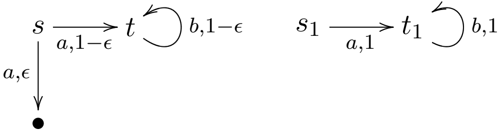
ϵ-simulation and ϵ-bisimulation

Definition

A relation $\mathcal{R} \subseteq S \times S$ is an ϵ-simulation if whenever $s\mathcal{R}t$, then $\forall a$, if $s \xrightarrow{a} \mu$, then $\exists t \xrightarrow{a} \nu$ such that for all $X \subseteq S$

$$\mu(X) \leq \nu(\mathcal{R}(X)) + \epsilon.$$

s is ϵ-simulated by t, written $s \prec_\epsilon t$, if $s\mathcal{R}t$ for some such $\mathcal{R}$.
If $\mathcal{R}$ is symmetric, it is an ϵ-bisimulation.



Then $s \prec_0 s_1$, $s_1 \not\prec_0 s$. and $s \not\sim_0 s_1$.

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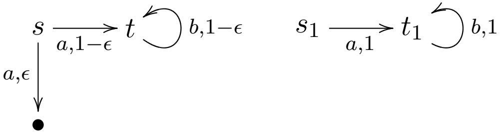

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The ϵ-semantics of logic $\mathcal{L}$.

Syntax:

$$\mathcal{L} \quad : \quad \theta ::= \top \mid \theta_1 \wedge \theta_2 \mid \theta_1 \vee \theta_2 \mid \langle a \rangle_\delta \, \theta, \text{ with } \delta \in [0;1]$$
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Semantics: let $\epsilon \in [-1;1]$

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Logical characterisations for fully probabilistic.

Definition (Logical simulation and bisimulation)

- $s \prec_\epsilon^{\mathcal{L}} t$ if for all $\theta \in \mathcal{L}$ we have $s \models \theta \Rightarrow t \models_\epsilon \theta$.
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Theorem

For fully probabilistic PAs

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Different metrics

- Approximate analysis of probabilistic processes: logic, simulation and games
  Desharnais, Laviolette, Tracol, Qest 08. Very good complexity

  Very different from others as probabilities are not multiplied through traces.

- Distances for Weighted Transition Systems: Games and Properties
  Fahrenberg, Thrane, Larsen QAPL '11.

- Testing Probabilistic Equivalence Through Reinforcement Learning
  Desharnais, Laviolette, Zhioua, FSTTCS '06.
  Very fast!!! and does not need the model

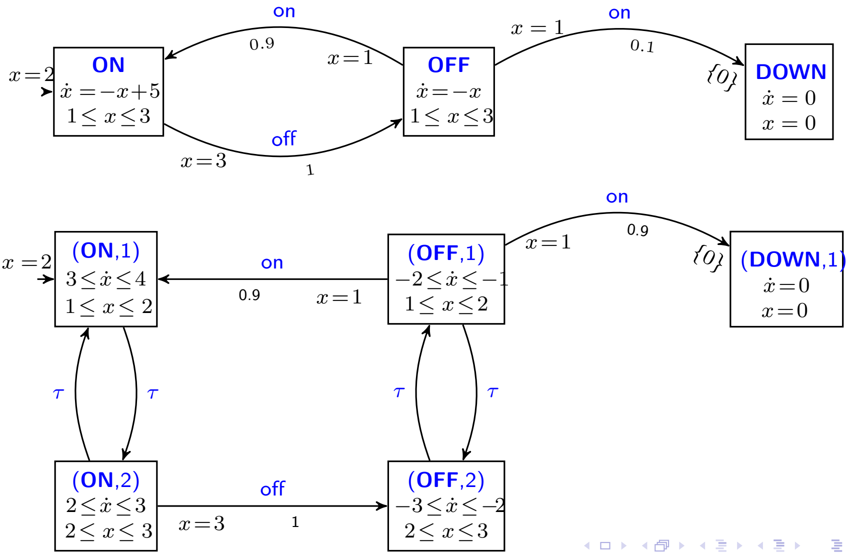
Approximation of probabilistic hybrid systems

- Analysis of Non-Linear Probabilistic Hybrid Systems, Desharnais, Assouramou,QAPL '11.
  - clock translation$\longrightarrow$ bisimilar timed automaton
  - linear phase-portrait approximation $\longrightarrow$ simulating rectangular HA
- Safety Verification for Probabilistic Hybrid Systems Zhang, She, Ratschan, Hermanns, Hahn, CAV '10.

Define a finite approximant or *abstraction* by quotienting, that over-approximate the original system.

Approximation of probabilistic hybrid systems [DA11]

A linear phase approx for some thermostat



Desirable properties of approximants? – The end

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Hybrid approximations are not constructed systematically but still satisfy some of these properties.

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