A Stochastic Broadcast Calculus

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1. Problems

- Markovian Arrival Process such as Batch Markovian Arrival Process (BMAP)[2] and Marked Markovian Arrival Process (MMAP)[1] are widely used in performance analysis.
- Examples such as customer arrivals in a supermarket and service requests for a server can be modeled by BMAP or MMAP.
- But usually we need to model systems in a compositional way where components can synchronize and interactive with each other.
- We also would like to get a CTMC from this model very naturally.

2. Example-Supermarket

- Suppose there are two types of customers, one is personal purchase (PP) and the other one is group purchase (GP).
- PP and GP come to cashiers in batch with different rates. For example we may say that the rate for 3 PPs and 2 GPs coming at the same time is 0.5.
- MMAP can be used to model customer arrivals easily, but if we want to model cashiers in this system and consider synchronization between customers and cashiers, then we need a more powerful tool.

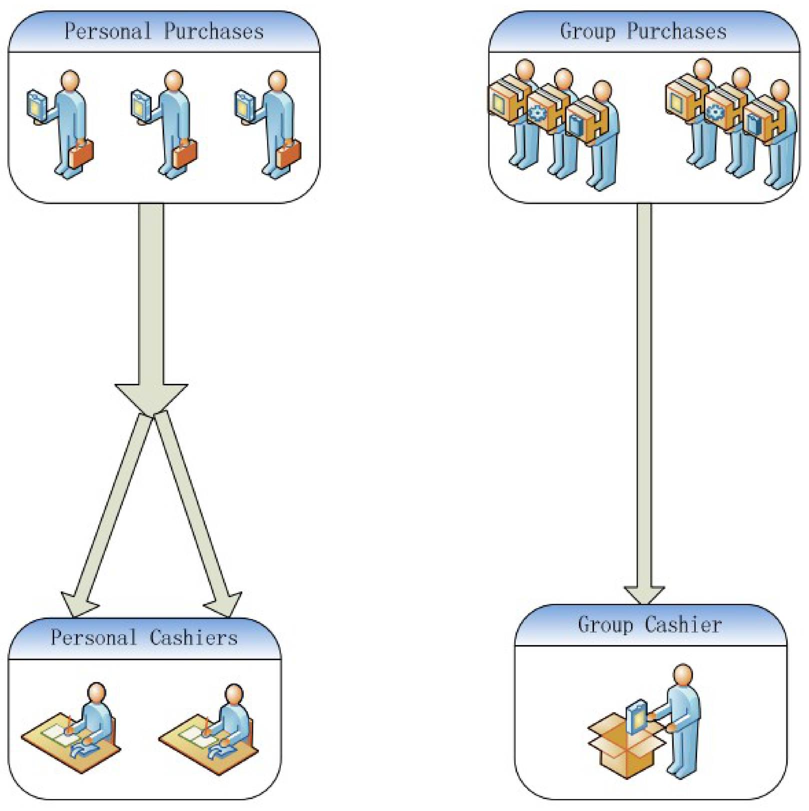

FIGURE 1: *Check out in a Supermarket*

3. Example-Closed Queueing Networks

- Closed Queueing Networks (CQN) [3] is another useful model in performance analysis.
- Customers must proceed from one server to another in order to satisfy their service requirements. After finished on a server, the customer will be transferred to one of the successors with certain probabilities.
- The queueing network is closed in the sense that neither arrivals nor departures of customers are permitted.

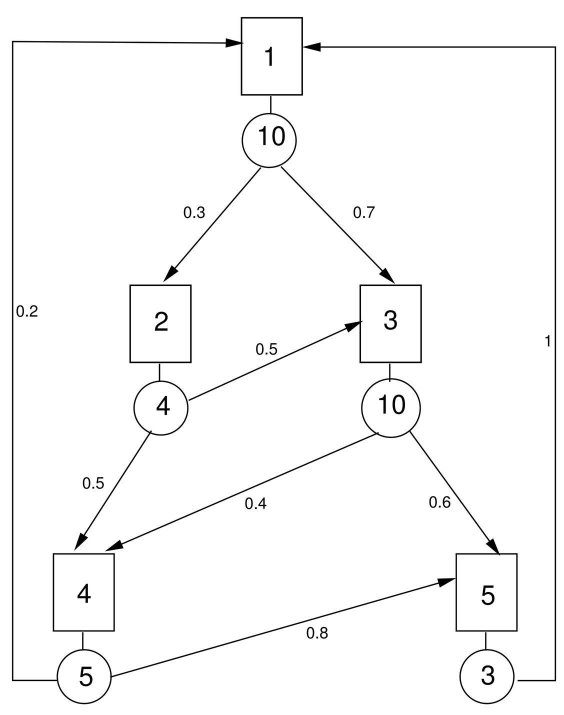

FIGURE 2: *A CQN with 5 Servers*

Figure 2 is a typical CQN with 5 servers and 15 customers where the numbers in the rectangles denote the length of queue of each server as well as their indexes, the numbers on the edges denote the transition probabilities and the numbers in the circles denote the rate of each service time.

We want that this CQN can be modeled in a compositional way especially when the probabilities are dynamically (depending on the number of the waiting customers for example).

4. Syntax

$$\mathsf{Act} ::= n(x,w) \mid n\langle m,\lambda\rangle$$

$$P, Q ::= 0 \mid \mathsf{Act}.P \mid \nu a P \mid P + Q \mid [n=m]P, Q \mid P \parallel Q \mid P \,\&\, Q \mid A$$

Every input $n(x,w)$ in our calculus is passive with a weight while every output $n\langle m,\lambda\rangle$ has a rate and non-blocked.

5. Semantics

We also give a Labeled Transition System for our calculus from which we can get a CTMC directly.

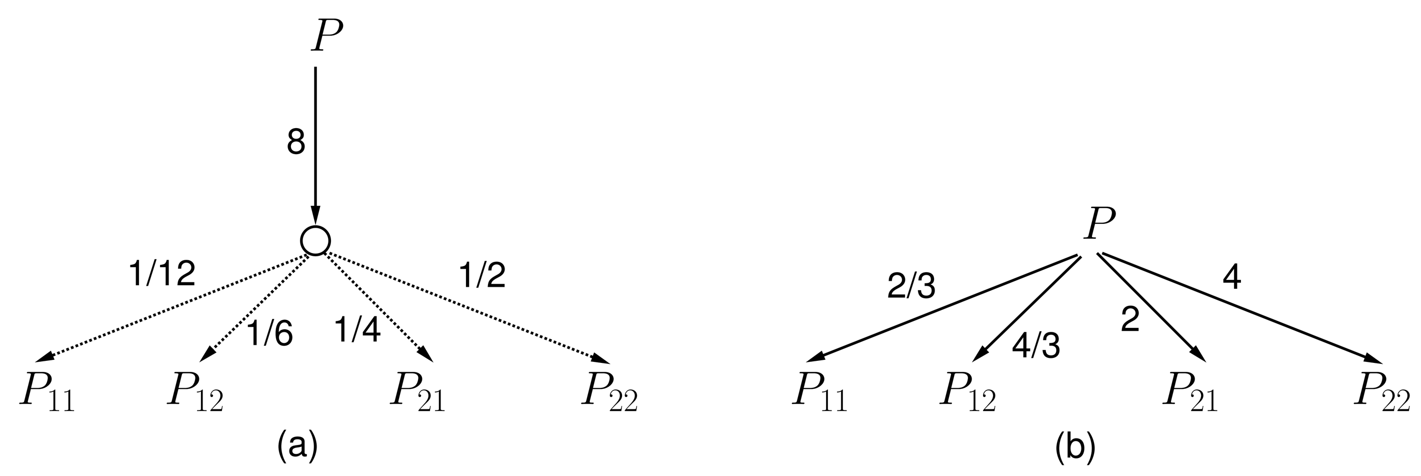

FIGURE 3: *A Simple CTMC*

$$\begin{aligned} P &\equiv a\langle b_1,2\rangle \parallel a\langle b_2,6\rangle \parallel (a(x,1).P_1 + a(x,2).P_2)\\ P_{11} &= a\langle b_2,6\rangle \parallel P_1\{b_1/x\} \quad P_{12} = a\langle b_2,6\rangle \parallel P_2\{b_1/x\}\\ P_{21} &= a\langle b_1,2\rangle \parallel P_1\{b_2/x\} \quad P_{22} = a\langle b_1,2\rangle \parallel P_2\{b_2/x\} \end{aligned}$$

The following example show the use of operator $\&$.
$Q \equiv a\langle b,3\rangle \parallel (a(x,1).Q_1 \,\&\, a(x,2).Q_2)$. Intuitively, Q can broadcast a message b on channel a with rate 3 and either Q_1 or Q_2 will receive it based on their weights but not both of them. So $Q \xrightarrow{3} \mathbb{Q}_a$ where

$$\mathbb{Q}_a \equiv \{\frac{1}{2+1} : (a\langle b\rangle, Q_1\{b/x\} \,\&\, a(x,2).Q_2), \frac{2}{2+1} : (a\langle b\rangle, a(x,1).Q_1 \,\&\, Q_2\{b/x\})\}.$$

6. Model of CQN

$\mathsf{SQ}_i(w_i)$ denotes the server i with i customers waiting for service.

$$\begin{aligned}
\mathsf{SQ}_1(w_1) &= c_4(x,0.2/(w_1+1)).[x=w](\mathsf{SQ}_1(w_1+1) \,\&\, \mathsf{SQ}_4(w-1))\\
&+ c_5(x,1/(w_1+1)).[x=w](\mathsf{SQ}_1(w_1+1) \,\&\, \mathsf{SQ}_5(w-1))\\
&+ c_1\langle w_1,10\rangle \quad w_1>0\\
\mathsf{SQ}_1(0) &= c_4(x,0.2).[x=w](\mathsf{SQ}_1(1) \,\&\, \mathsf{SQ}_4(w-1))\\
&+ c_5(x,1).[x=w](\mathsf{SQ}_1(w_1+1) \,\&\, \mathsf{SQ}_5(w-1))\\
\mathsf{SQ}_2(w_2) &= c_1(x,0.3/(w_2+1)).[x=w](\mathsf{SQ}_2(w_2+1) \,\&\, \mathsf{SQ}_1(w_1-1))\\
&+ c_2\langle s,4\rangle \quad w_2>0\\
\mathsf{SQ}_2(0) &= c_1(x,0.3).[x=w](\mathsf{SQ}_2(1) \,\&\, \mathsf{SQ}_1(w_1-1))\\
\mathsf{SQ}_3(w_3) &= c_1(x,0.7/(w_3+1)).[x=w](\mathsf{SQ}_3(w_3+1) \,\&\, \mathsf{SQ}_1(w-1))\\
&+ c_2(x,0.5/(w_3+1)).[x=w](\mathsf{SQ}_3(w_3+1) \,\&\, \mathsf{SQ}_2(w-1))\\
&+ c_3\langle s,10\rangle \quad w_3>0\\
\mathsf{SQ}_3(0) &= c_1(x,0.7).[x=w](\mathsf{SQ}_3(1) \,\&\, \mathsf{SQ}_1(w-1))\\
&+ c_2(x,0.5).[x=w](\mathsf{SQ}_3(1) \,\&\, \mathsf{SQ}_2(w-1))\\
\mathsf{SQ}_4(w_4) &= c_2(x,0.5/(w_4+1)).[x=w](\mathsf{SQ}_4(w_4+1) \,\&\, \mathsf{SQ}_2(w-1))\\
&+ c_3(x,0.4/(w_4+1)).[x=w](\mathsf{SQ}_4(w_4+1) \,\&\, \mathsf{SQ}_3(w-1))\\
&+ c_4\langle s,5\rangle \quad w_4>0\\
\mathsf{SQ}_4(0) &= c_2(x,0.5).[x=w](\mathsf{SQ}_4(1) \,\&\, \mathsf{SQ}_2(w-1))\\
&+ c_3(x,0.4).[x=w](\mathsf{SQ}_4(1) \,\&\, \mathsf{SQ}_3(w-1))\\
\mathsf{SQ}_5(w_5) &= c_3(x,0.6/(w_5+1)).[x=w](\mathsf{SQ}_5(w_5+1) \,\&\, \mathsf{SQ}_3(w-1))\\
&+ c_4(x,0.8/(w_5+1)).[x=w](\mathsf{SQ}_5(w_5+1) \,\&\, \mathsf{SQ}_4(w-1))\\
&+ c_5\langle s,3\rangle \quad w_5>0\\
\mathsf{SQ}_5(0) &= c_3(x,0.6).[x=w](\mathsf{SQ}_5(1) \,\&\, \mathsf{SQ}_3(w-1))\\
&+ c_4(x,0.8).[x=w](\mathsf{SQ}_5(1) \,\&\, \mathsf{SQ}_4(w-1))
\end{aligned}$$

The system in Figure 2 can be denoted as $\&_{i=1}^{5}\mathsf{SQ}_i(i)$.

References

[1] Q.M. HE and M.F. NEUTS. Markov chains with marked transitions, *Stochastic processes and their applications*, 74(1):37-52,1998. Elsevier Science.

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[3] Buzen, J.P. Computational Algorithms for Closed Queueing Networks with Exponential Servers, *Communications of the ACM*, 16(9):527–531,1973. ACM.