ODE and Discrete Simulation or Mean Field Methods for Computer and Communication Systems

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1

MEAN FIELD INTERACTION MODEL

Mean Field

- A *model* introduced in Physics
  - interaction between *particles* is via distribution of states of all particle
- An *approximation* method for a large collection of particles
  - assumes *independence* in the master equation
- Why do we care in I&C?
  - Model interaction of many objects:
  - Distributed systems, communication protocols, game theory, self-organized systems

Mean Field Interaction Model

- Time is discrete

- N objects, N large
- Object n has state $X_n(t)$
- $(X^N_1(t), \ldots, X^N_N(t))$ is Markov

- Objects are observable only through their state

- "Occupancy measure"
  $M^N(t)$ = distribution of object states at time t

Mean Field Interaction Model

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- Objects are observable only through their state

- "Occupancy measure" $M^N(t)$ = distribution of object states at time t

- ***Theorem*** [Gast (2011)] $M^N(t)$ is Markov

- Called "*Mean Field Interaction Models*" in the Performance Evaluation community [McDonald(2007), Benaïm and Le Boudec(2008)]

A Few Examples Where Applied

[1] L. Afanassieva, S. Popov, and G. Fayolle. Models for transportation networks. *Journal of Mathematical Sciences, 1997 – Springer*.

[2] F. Baccelli, A. Chaintreau, D. De Vleeschauwer, and D. R. McDonald. Http turbulence, May 2004.

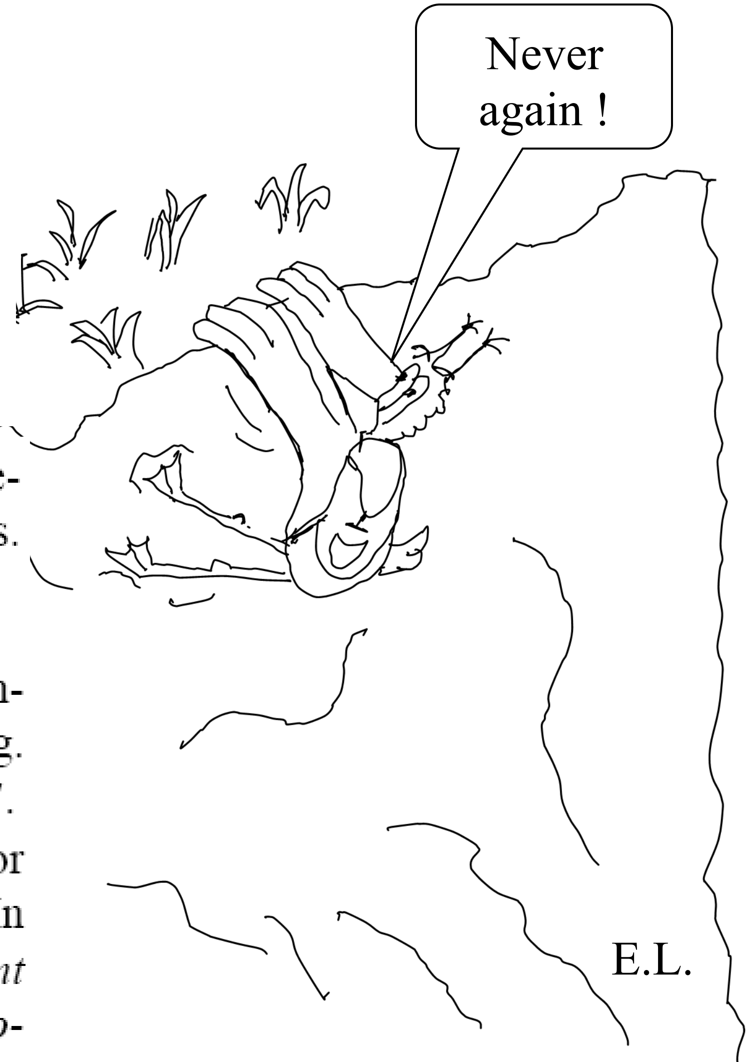
[3] F. Baccelli, M. Lelarge, and D. McDonald. Metastable regimes for multiplexed tcp flows. In *Proceedings of the*

[5] M.-D. Bordenave, Charles and A. Proutiere. A particle system in interaction with a rapidly varying environment: Mean field limits and applications. arXiv:math/0701363v2.

[11] S. Kumar and L. Massoulié. Integrating streaming and file-transfer internet traffic: Fluid and diffusion approximations. MSR-TR-2005-160.

[16] Y. M. Suhov and N. D. Vvedenskaya. Dobrushin's mean-field approximation for a queue with dynamic routing. *Markov Processes and Related Fields*, 3(4):493–526, 1997.

[17] P. Tinnakornsrisuphap and A. M. Makowski. Limit behavior of ecn/red gateways under a large number of tcp flows. In *Proceedings IEEE INFOCOM 2003, The 22nd Annual Joint Conference of the IEEE Computer and Communications Societies, San Franciso, CA, USA*, March 30 - April 3 2003.



Example: 2-Step Malware

- Mobile nodes are either
  - `S' Susceptible
  - `D' Dormant
  - `A' Active
- Time is discrete
- Nodes meet pairwise (bluetooth)
- One interaction per time slot,
  I(N) = 1/N; mean field limit is an ODE

- State space is finite
  = {`S' , `A' ,`D'}

- Occupancy measure is
  M(t) = (S(t), D(t), A(t)) with
  S(t)+ D(t) + A(t) =1

S(t) = proportion of nodes in state `S'

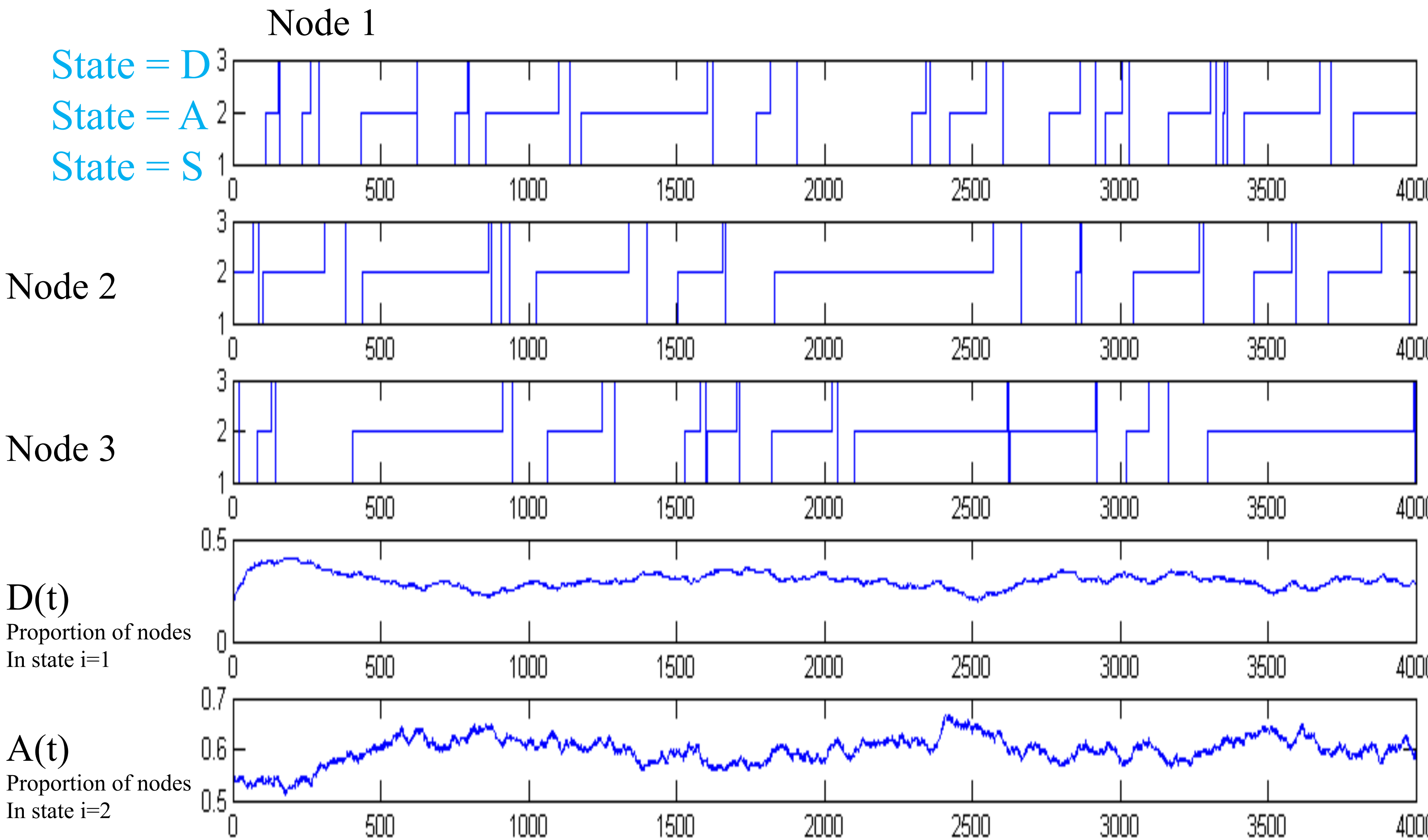
[Benaïm and Le Boudec(2008)]

- Possible interactions:

1. Recovery
   - D -> S
2. Mutual upgrade
   - D + D -> A + A
3. Infection by active
   - D + A -> A + A
4. Recovery
   - A -> S
5. Recruitment by Dormant
   - S + D -> D + D

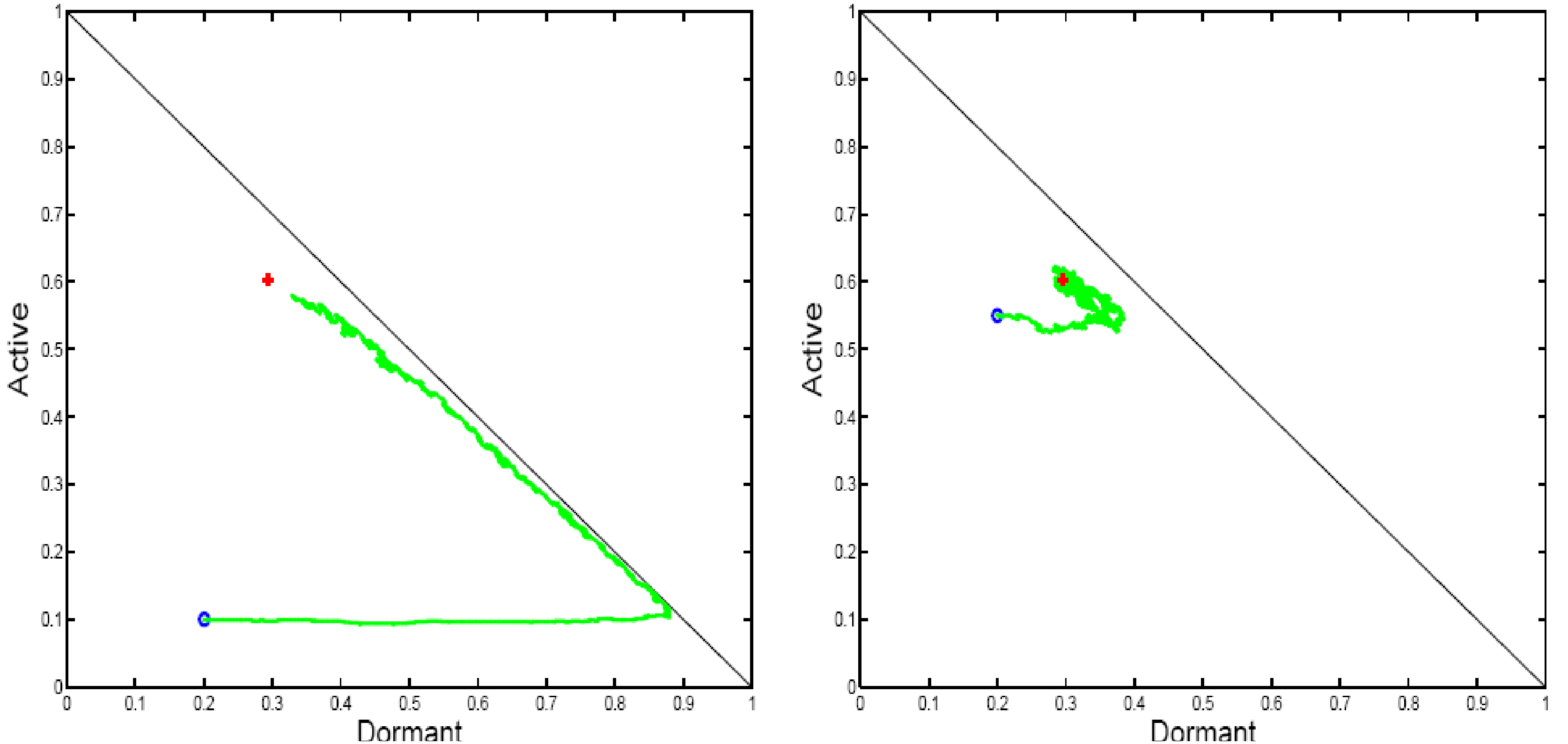
   Direct infection
   - S -> D
6. Direct infection
   - S -> A

Simulation Runs, N=1000 nodes



$$\beta = 0.01, \delta_A = 0.005, \delta_D = 0.0001, \alpha_0 = \alpha = 0.0001, h = 0.3, r = 0.1, \lambda = 0.0001$$

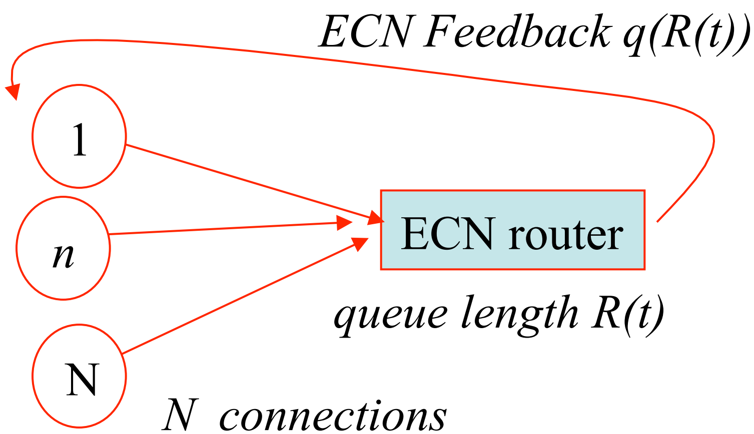
Sample Runs with N = 1000



$$\beta = 0.01, \delta_A = 0.005, \delta_D = 0.0001, \alpha_0 = \alpha = 0.0001, h = 0.3, r = 0.1, \lambda = 0.0001$$

Example: TCP and ECN

- [Tinnakornsrisuphap and Makowski(2003)]



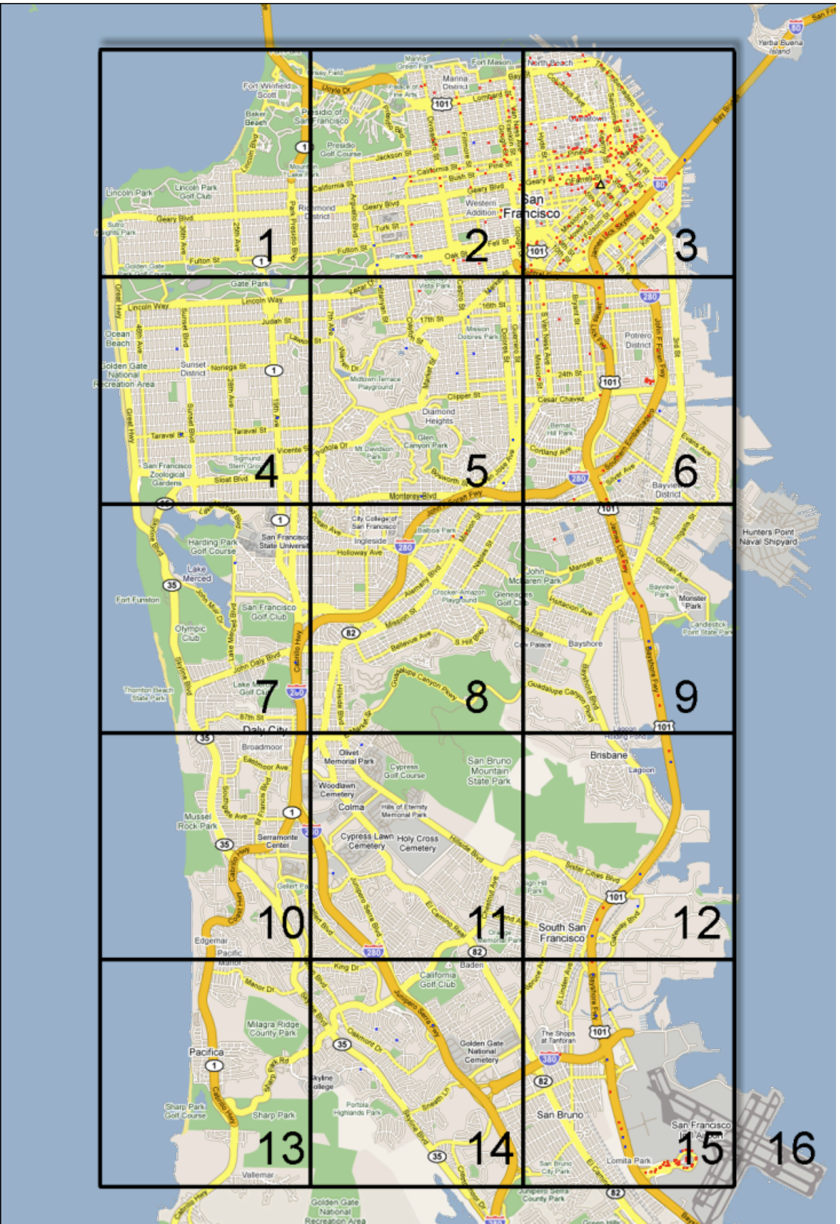
At, every time step, all connections update their state: I(N)=1

- Time is discrete, mean field limit is also in discrete time (iterated map)

- Similar examples:
  HTTP Metastability [Baccelli et al.(2004)Baccelli, Lelarge, and McDonald]

  Reputation System [Le Boudec et al.(2007)Le Boudec, McDonald, and Mundinger]

The Importance of Being Spatial

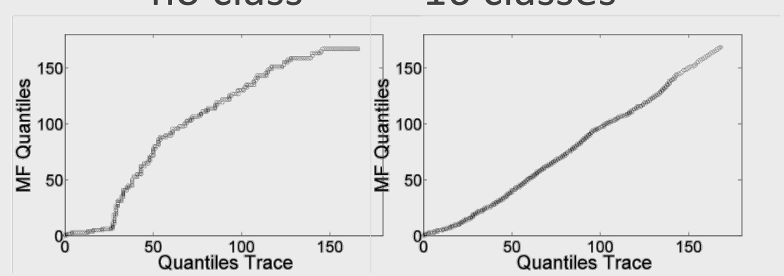


- Mobile node state = (c, t)
  c = 1 … 16 (position)
  $t \in R^+$ (age of gossip)

- Time is continuous, I(N) = 1
- Occupancy measure is
  $F_c(z,t)$ = proportion of nodes that at location *c* and have age ≤ *z*

  [Age of Gossip, Chaintreau et al. (2009)]

no class 16 classes

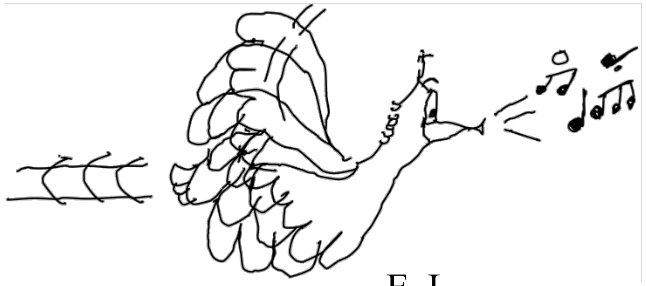


Qqplots simulation vs mean field

What can we do with a Mean Field Interaction Model ?

- Large N asymptotics, Finite Horizon
  - fluid limit of occupancy measure (ODE)
  - decoupling assumption (fast simulation)
- Issues
  - When valid
  - How to formulate the fluid limit

- Large t asymptotic
  - Stationary approximation of occupancy measure
  - Decoupling assumption
- Issues
  - When valid



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2.

CONVERGENCE TO ODE

Intensity *I(N)*

- $I(N)$ = expected number of transitions per object per time unit

- A mean field limit occurs when we re-scale time by $I(N)$
  i.e. we consider $X^N(t/I(N))$

- I(N) = O(1): mean field limit is in discrete time
  [Le Boudec et al (2007)]

  I(N) = O(1/N): mean field limit is in continuous time
  [Benaïm and Le Boudec (2008)]

The Mean Field Limit

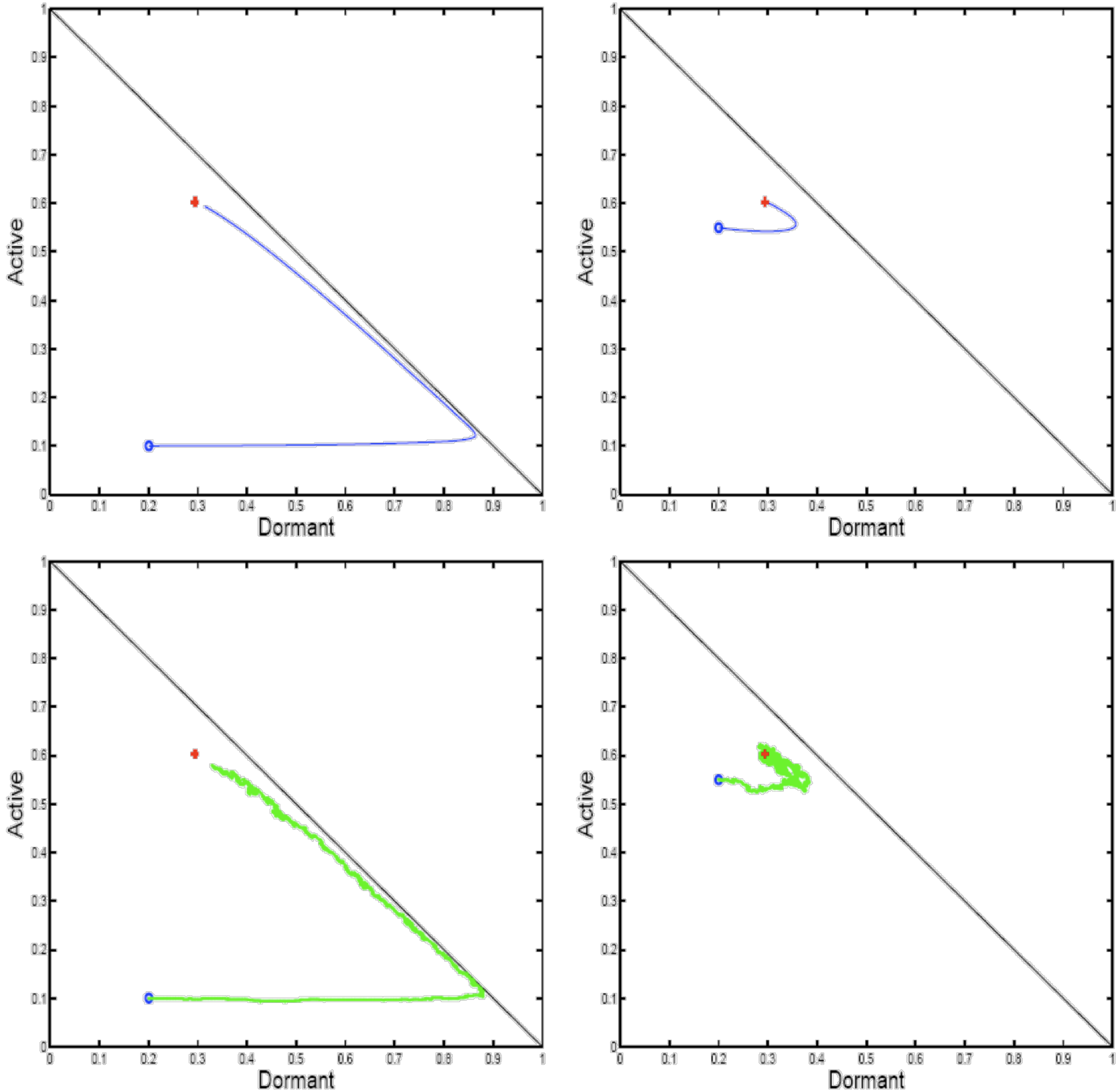
- Under very general conditions (given later) the occupancy measure converges, in law, to a deterministic process, *m(t),* called the *mean field limit*

$$M^N\left(\frac{t}{I(N)}\right) \to m(t)$$

- Finite State Space => ODE

Mean Field Limit
N = +∞

Stochastic system
N = 1000



Sufficient Conditions for Convergence

- [Kurtz 1970], see also [Bordenav et al 2008], [Graham 2000]
- Sufficient conditon verifiable by inspection:

  [Benaïm and Le Boudec(2008), Ioannidis and Marbach(2009)]

  - Let $W^N(k)$ be the number of objects that do a transition in time slot k. Note that $\mathbb{E}\left(W^N(k)\right) = NI(N)$, where $I(N) \stackrel{\text{def.}}{=}$ intensity. Assume

  $$\mathbb{E}\left(W^N(k)^2\right) \leq \beta(N) \quad \text{with} \quad \lim_{N\to\infty} I(N)\beta(N) = 0$$

  Example: I(N) = 1/N
  Second moment of number of objects
  affected in one timeslot = o(N)

- Similar result when mean field limit is in discrete time [Le Boudec et al 2007]

Example: Convergence to Mean Field

Example: 2-Step Malware

- Mobile nodes are either
  - `S' Susceptible
  - `D' Dormant
  - `A' Active
- Time is discrete
- Nodes meet pairwise (bluetooth)
- One interaction per time slot, I(N) = 1/N; mean field limit is an ODE
- State space is finite = {`S' , `A' ,`D'}
- Occupancy measure is M(t) = (S(t), D(t), A(t)) with S(t)+ D(t) + A(t) =1

S(t) = proportion of nodes in state `S'

[Benaïm and Le Boudec(2008)]

- Possible interactions:

1. Recovery
   - D -> S
2. Mutual upgrade
   - D + D -> A + A
3. Infection by active
   - D + A -> A + A
4. Recovery
   - A -> S
5. Recruitment by Dormant
   - S + D -> D + D

   Direct infection
   - S -> D
6. Direct infection
   - S -> A

7

- Rescale time such that one time step = 1/N
- Number of transitions per time step is bounded by 2, therefore there is convergence to mean field

$$\frac{\partial D}{\partial t} = -\delta_D D - 2\lambda D^2 - \beta A \frac{D}{h+D} + (\alpha_0 + rD)S$$

$$\frac{\partial A}{\partial t} = 2\lambda D^2 + \beta A \frac{D}{h+D} - \delta_A A + \alpha S$$

$$\frac{\partial S}{\partial t} = \delta_D D + \delta_A A - (\alpha_0 + rD)S - \alpha S$$

Formulating the Mean Field Limit

- *Drift* = sum over all transitions of

  proba of transition

  x

  Delta to system state $M^N(t)$

- Re-scale drift by intensity

- Equation for mean field limit is

  dm/dt = limit of rescaled drift

- Can be automated

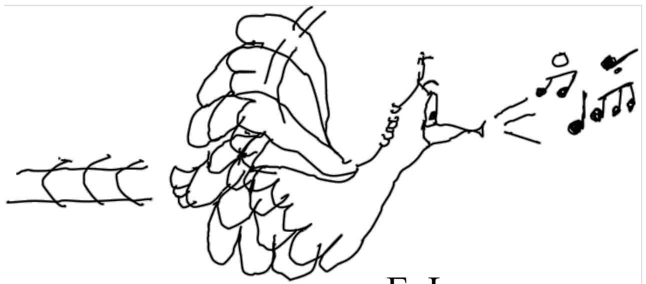
  http://icawww1.epfl.ch/IS/tsed

case	prob	effect on (D, A, S)
1	$D\delta_D$	$\frac{1}{N}(-1, 0, 1)$
2	$D\lambda\frac{ND-1}{N-1}$	$\frac{1}{N}(-2, +2, 0)$
3	$A\beta\frac{D}{h+D}$	$\frac{1}{N}(-1, +1, 0)$
4	$A\delta_A$	$\frac{1}{N}(0, -1, +1)$
5	$S(\alpha_0 + rD)$	$\frac{1}{N}(+1, 0, -1)$
6	$S\alpha$	$\frac{1}{N}(0, +1, -1)$

$$\text{drift} = \frac{1}{N}\begin{pmatrix} -D\delta_D - 2D\lambda\frac{ND-1}{N-1} - A\beta\frac{D}{h+D} + S(\alpha_0 + rD) \\ 2D\lambda\frac{ND-1}{N-1} + A\beta\frac{D}{h+D} - A\delta_A + S\alpha \\ D\delta_D + A\delta_A - S(\alpha_0 + rD) - S\alpha \end{pmatrix}$$

$$\begin{aligned} \frac{\partial D}{\partial t} &= -\delta_D D - 2\lambda D^2 - \beta A\frac{D}{h+D} + (\alpha_0 + rD)S \\ \frac{\partial A}{\partial t} &= 2\lambda D^2 + \beta A\frac{D}{h+D} - \delta_A A + \alpha S \\ \frac{\partial S}{\partial t} &= \delta_D D + \delta_A A - (\alpha_0 + rD)S - \alpha S \end{aligned}$$

Convergence to Mean Field



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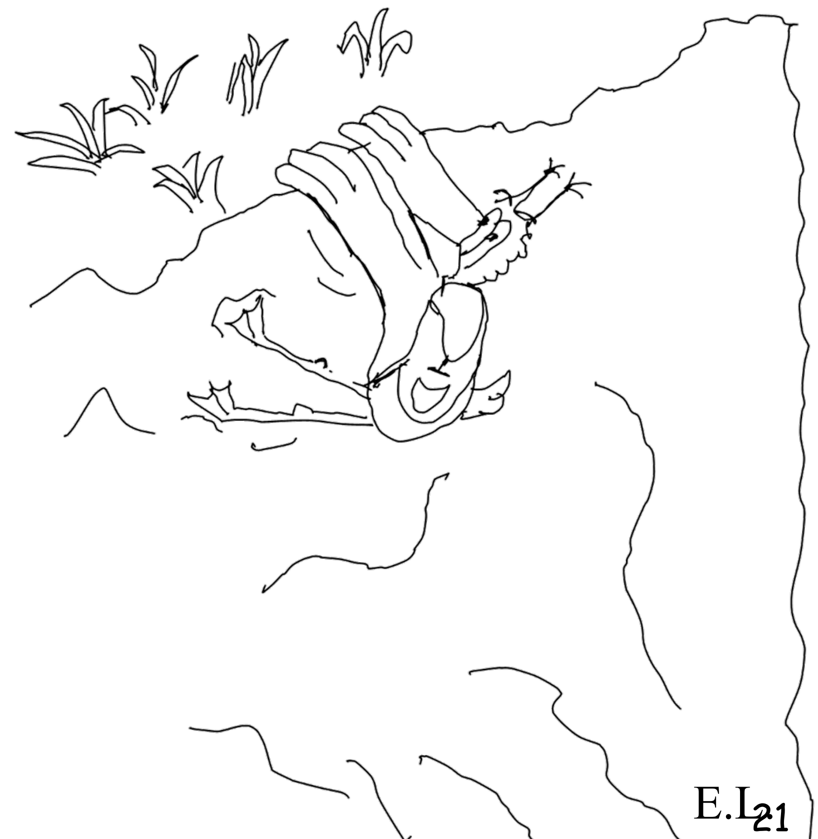
- For the general state space, things may be more complex (fluid limitz is not an ODE, e.g. [Chaintreau et al, 2009])

- For the finite state space case, there are many simple results, often verifiable by inspection

  For example [Kurtz 1970] or [Benaim, Le Boudec 2008]



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3.

FINITE HORIZON : FAST SIMULATION AND DECOUPLING ASSUMPTION

Convergence to Mean Field Limit is Equivalent to Propagation of Chaos

Definition 1.1 *Let* $X^N = (X_1^N, ..., X_N^N)$ *be an exchangeable sequence of processes in* $\mathcal{P}(S)$ *and* $m \in \mathcal{P}(S)$ *where* S *is metric complete separable.* $(X^N)_N$ *is* m*-chaotic iff for every* k*:* $\mathcal{L}(X_1^N, .., X_k^N) \to m \otimes \ldots \otimes m$ *as* $N \to \infty$.

Theorem 1.1 ([Sznitman(1991)]) $(X^N)_N$ *is* m*-chaotic then the occupancy measure* $M^N \stackrel{\text{def}}{=} \frac{1}{N} \sum_{n=1}^{N} \delta_{X_n^N}$ *converges in probability (and in law) to* m.

If the occupancy measure converges in law to m *then* $(X^N)_N$ *is* m*-chaotic.*

Propagation of Chaos = Decoupling Assumption

- **(Propagation of Chaos)**

  k objects are asymptotically independent with common law equal to the mean field limit, for any fixed *k*

$$\mathcal{L}\left(X_1\left(\frac{t}{I(N)}\right),\ldots,X_k\left(\frac{t}{I(N)}\right)\right) \to m(t)\otimes\ldots\otimes m(t)$$

- **(Decoupling Assumption)**
  (also called Mean Field Approximation, or Fast Simulation)
  The law of one object is asymptotically as if all other objects were drawn randomly with replacement from *m(t)*

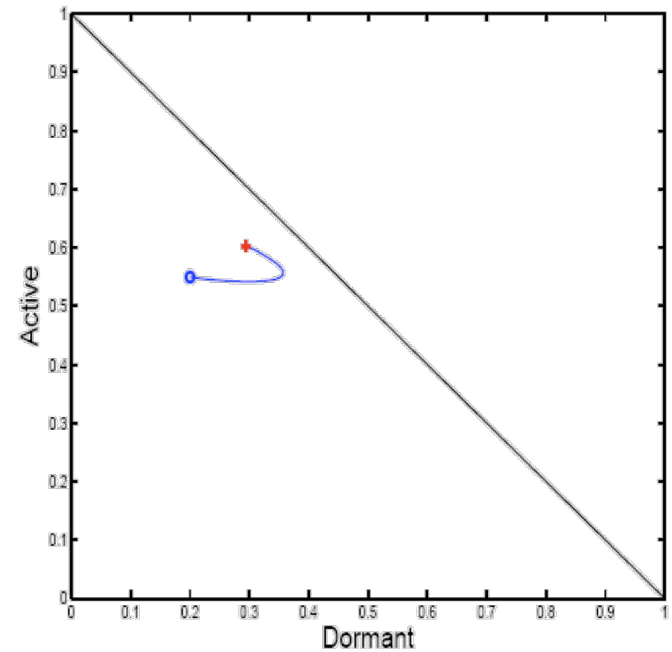
The Two Interpretations of the Mean Field Limit

- At any time t

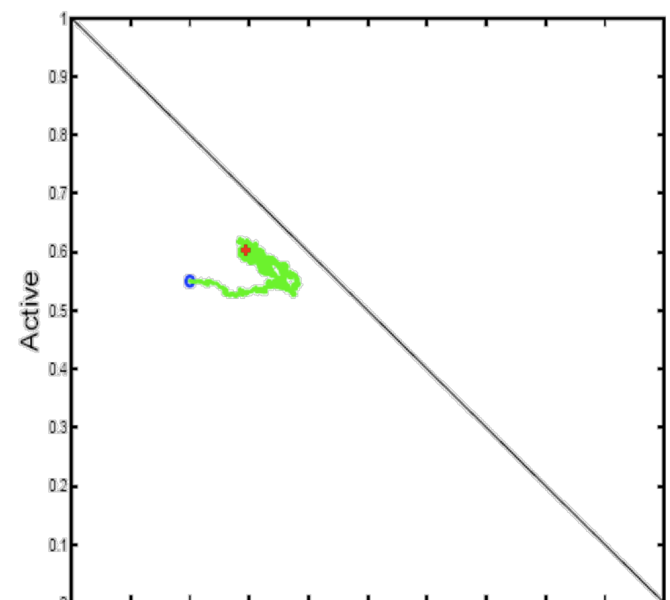
$$P(X_n(t) = 'A') \approx A\left(\frac{t}{N}\right)$$

$$P(X_m(t) = 'D', X_n(t) = 'A') \approx D\left(\frac{t}{N}\right) A\left(\frac{t}{N}\right)$$

where (D, A, S) is solution of ODE



- Thus for large t :
  - Prob (node n is dormant) ≈ 0.3
  - Prob (node n is active) ≈ 0.6
  - Prob (node n is susceptible) ≈ 0.1



- *m(t)* approximates both

1. the occupancy measure $M^N(t)$
2. the state probability for one object at time *t*, drawn at random among *N*

« Fast Simulation »

- $p^N_j(t|i)$ is the probability that a node that starts in state i is in state j at time t:

$$p_j^N(t|i) = \mathbb{P}(X_n^N(t) = j | X_n^N(0) = i)$$

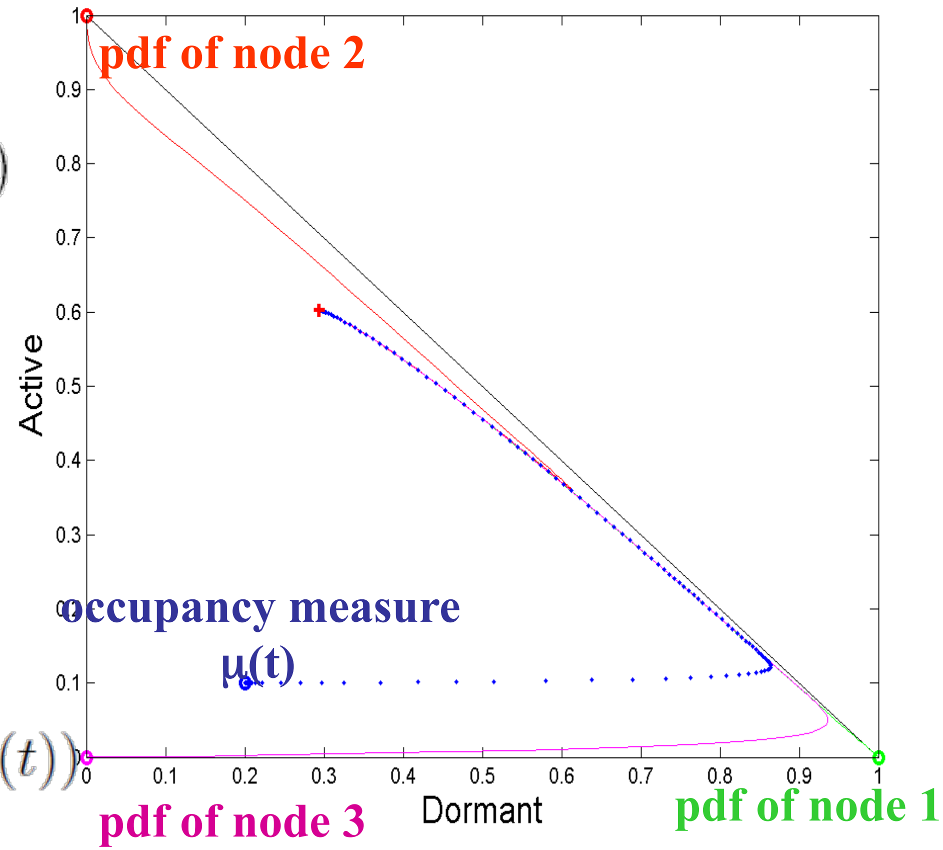
- Then $p_j^N(t/N|i) \approx p_j(t|i)$

  where *p(t|i)* is a continuous time, non homogeneous process

$$\frac{d}{dt}\vec{p}(t|i) = \vec{p}(t|i)^T A(\vec{m}(t)))$$

$$\frac{d}{dt}\vec{m}(t) = \vec{m}(t)^T A(\vec{m}(t))) = F(\vec{m}(t))$$

- Same ODE as mean field limit, but with different initial condition



The Decoupling Assumption

- The evolution for one object as if the other objects had a state drawn randomly and independently from the distribution *m(t)*

- Is valid *over finite horizon* whenever mean field convergence occurs

- Can be used to analyze or simulate evolution of *k* objects

4.

INFINITE HORIZON: FIXED POINT METHOD AND DECOUPLING ASSUMPTION

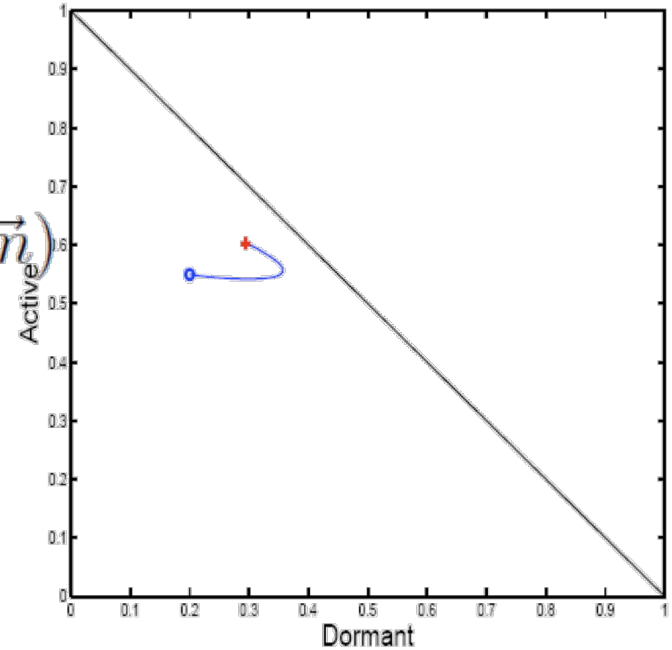
The Fixed Point Method

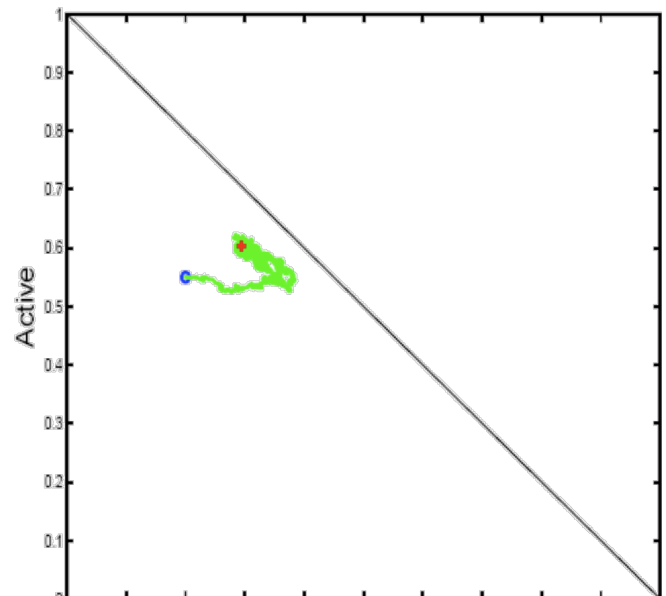
- Decoupling assumption says distribution of prob for state of one object is approx. *m(t)* with $\frac{d\vec{m}}{dt} = F(\vec{m})$

- We are interested in stationary regime, i.e we do $F(\vec{m}) = \vec{0}$

- This is the « Fixed Point Method »
- Example: in stationary regime:
  - Prob (node *n* is dormant) ≈ 0.3
  - Prob (node *n* is active) ≈ 0.6
  - Prob (node *n* is susceptible) ≈ 0.1

  - Nodes *m* and *n* are independent





Example: 802.11 Analysis, Bianchi's Formula

802.11 single cell

m_i = proba one node is in backoff stage I

β= attempt rate

γ = collision proba

See [Benaim and Le Boudec , 2008] for this analysis

ODE for mean field limit

$$\frac{dm_0}{d\tau} = -m_0 q_0 + \beta(\vec{m})\left(1-\gamma(\vec{m})\right) + q_K m_K \gamma(\vec{m})$$

$$\frac{dm_i}{d\tau} = -m_i q_i + m_{i-1} q_{i-1} \gamma(\vec{m}) \qquad i = 1, ..., K$$

$$\beta(\vec{m}) = \sum_{i=0}^{K} q_i m_i$$

$$\gamma(\vec{m}) = 1 - e^{-\beta(\vec{m})}$$

Solve for Fixed Point:

$$m_i = \frac{\gamma^i}{q_i} \frac{1}{\sum_{k=0}^{K} \frac{\gamma^k}{q_k}}$$

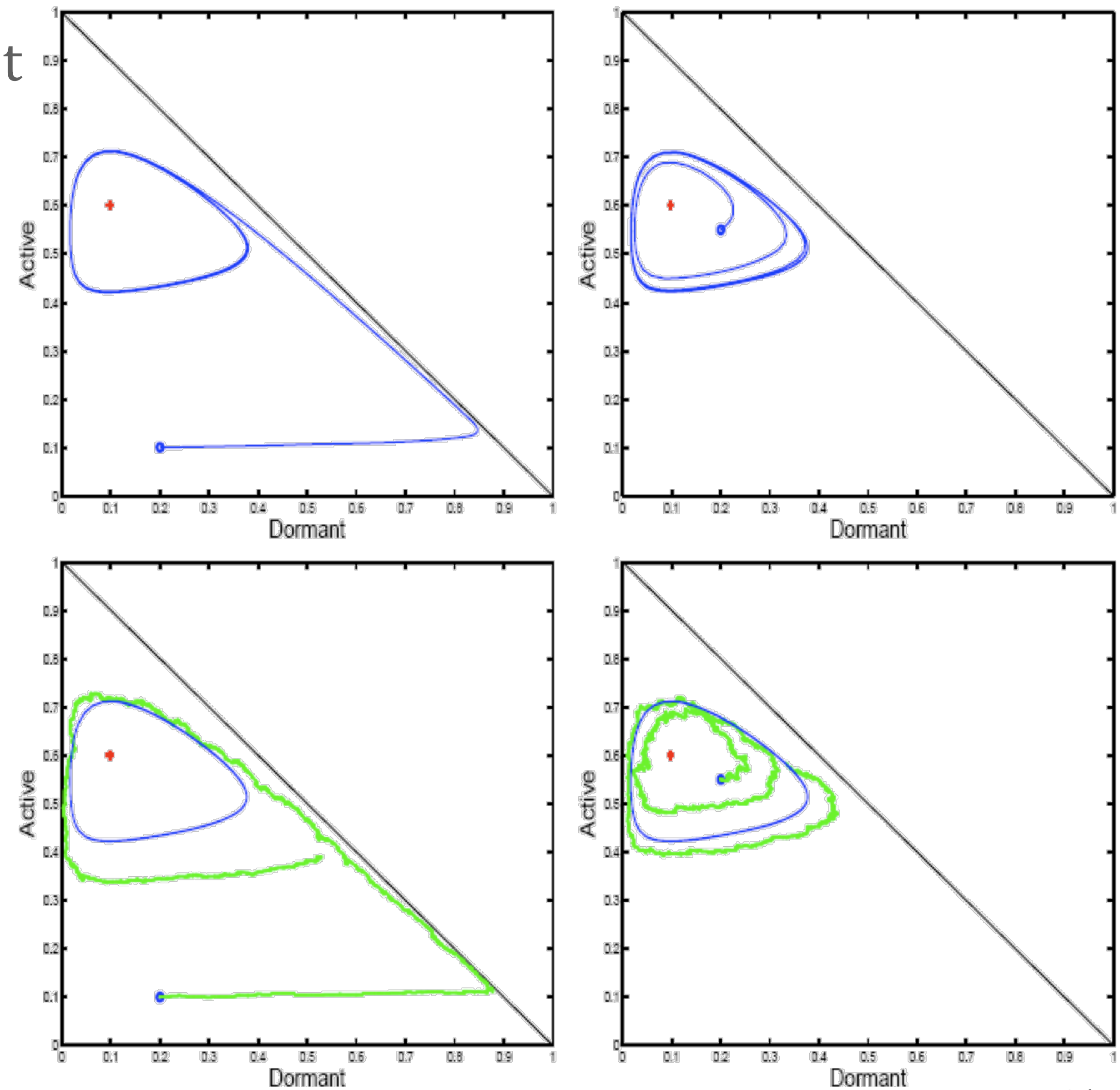
Bianchi's Fixed Point Equation [Bianchi 1998]

$$\gamma = 1 - e^{-\beta}$$

$$\beta = \frac{\sum_{k=0}^{K} \gamma^k}{\sum_{k=0}^{K} \frac{\gamma^k}{q_k}}$$

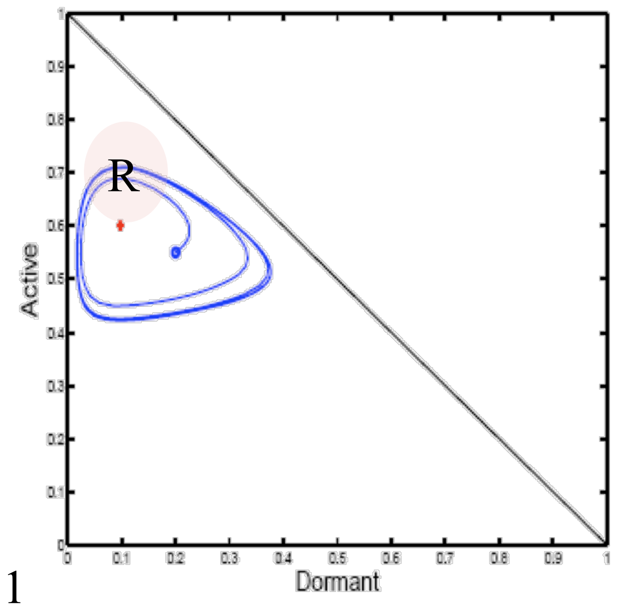
2-Step Malware, Again

- Same as before except for one parameter value : $h = 0.1$ instead of 0.3
- The ODE does not converge to a unique attractor (limit cycle)
- The equation $F(m) = 0$ has a unique solution (red cross) – but it is *not* the stationary regime !

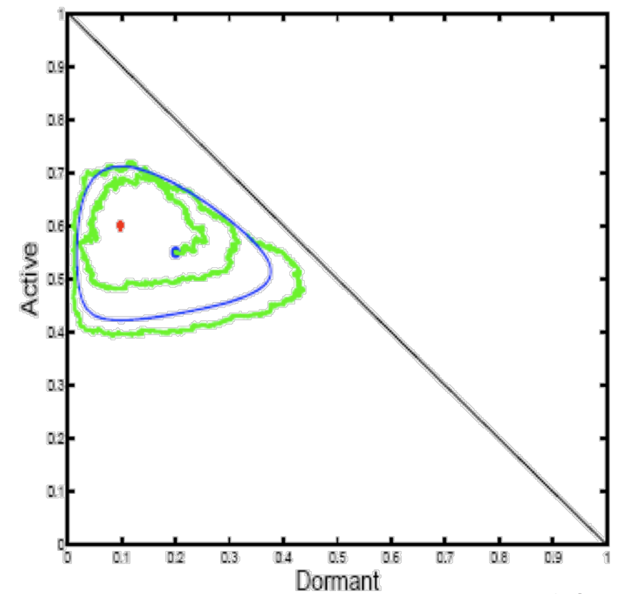


Example Where Fixed Point Method Fails

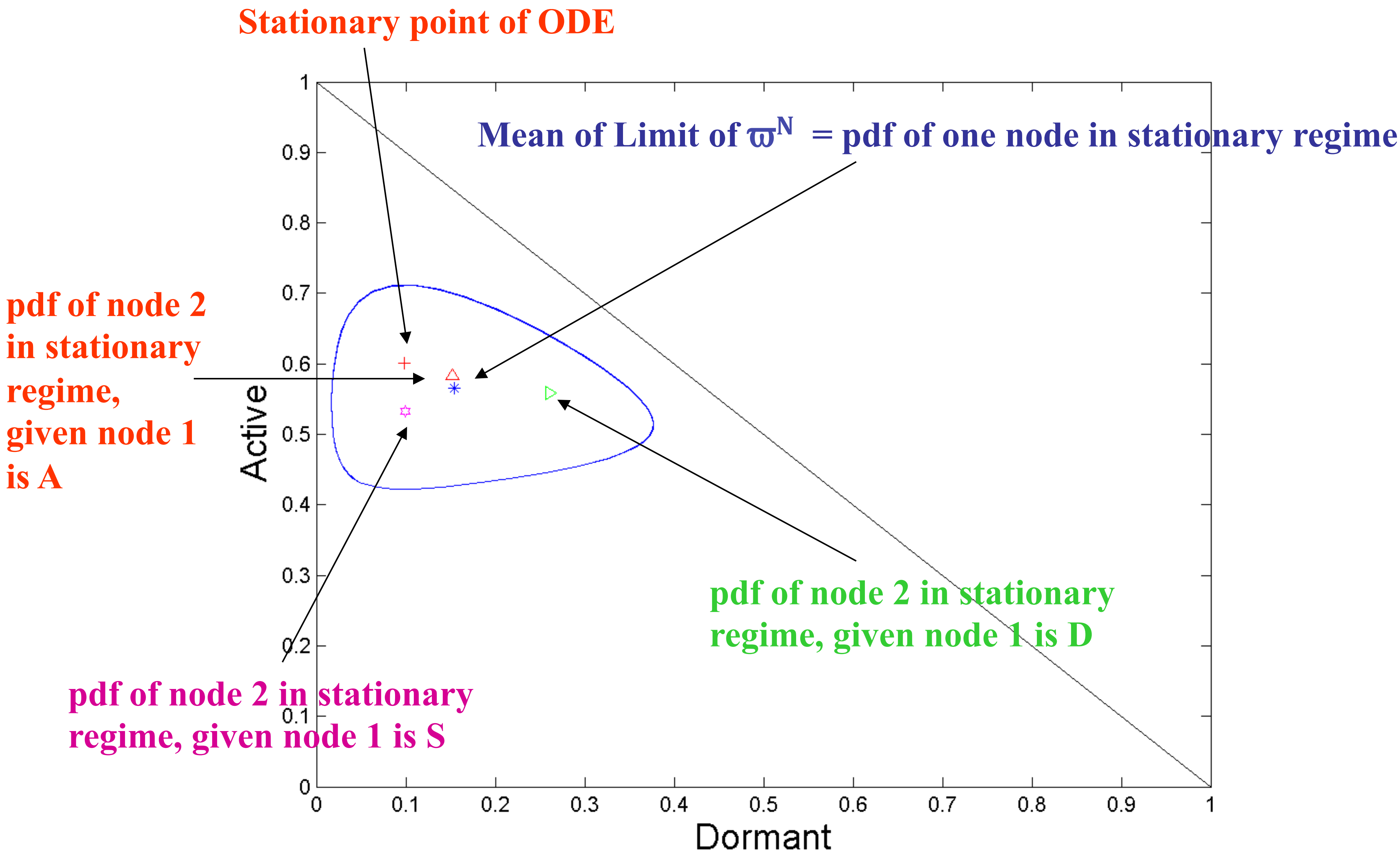
- In stationary regime, m(t) = (D(t), A(t), S (t)) follows the limit cycle
- Assume you are in stationary regime (simulation has run for a long time) and you observe that one node, say n=1, is in state 'A'
- It is more likely that m(t) is in region R
- Therefore, it is more likely that some other node, say n=2, is also in state 'A'
- This is synchronization



h=0.1



Joint PDFs of Two Nodes in Stationary Regime



Where is the Catch ?

- Decoupling assumption says that nodes m and n are asymptotically independent

- There *is* mean field convergence for this example

- But we saw that nodes may not be asymptotically independent

… is there a contradiction ?

Markov chain is ergodic

$$\mathbb{P}(X_1^N(t/N) = i \text{ and } X_1^N(t/N) = j) \xrightarrow{t\to\infty} \pi_{i,j}^N$$

$N\to\infty$ ↓ Mean Field Convergence | ↓ $N\to\infty$

$$m_i(t)\, m_j(t) \quad \neq \quad \frac{1}{T}\int_0^T m_i(t)\, m_j(t)\, dt$$

- The *decoupling assumption may not hold in stationary regime*, even for perfectly regular models

Result 1: Fixed Point Method Holds under (H)

- Assume that

(H) ODE has a unique global stable point to which all trajectories converge

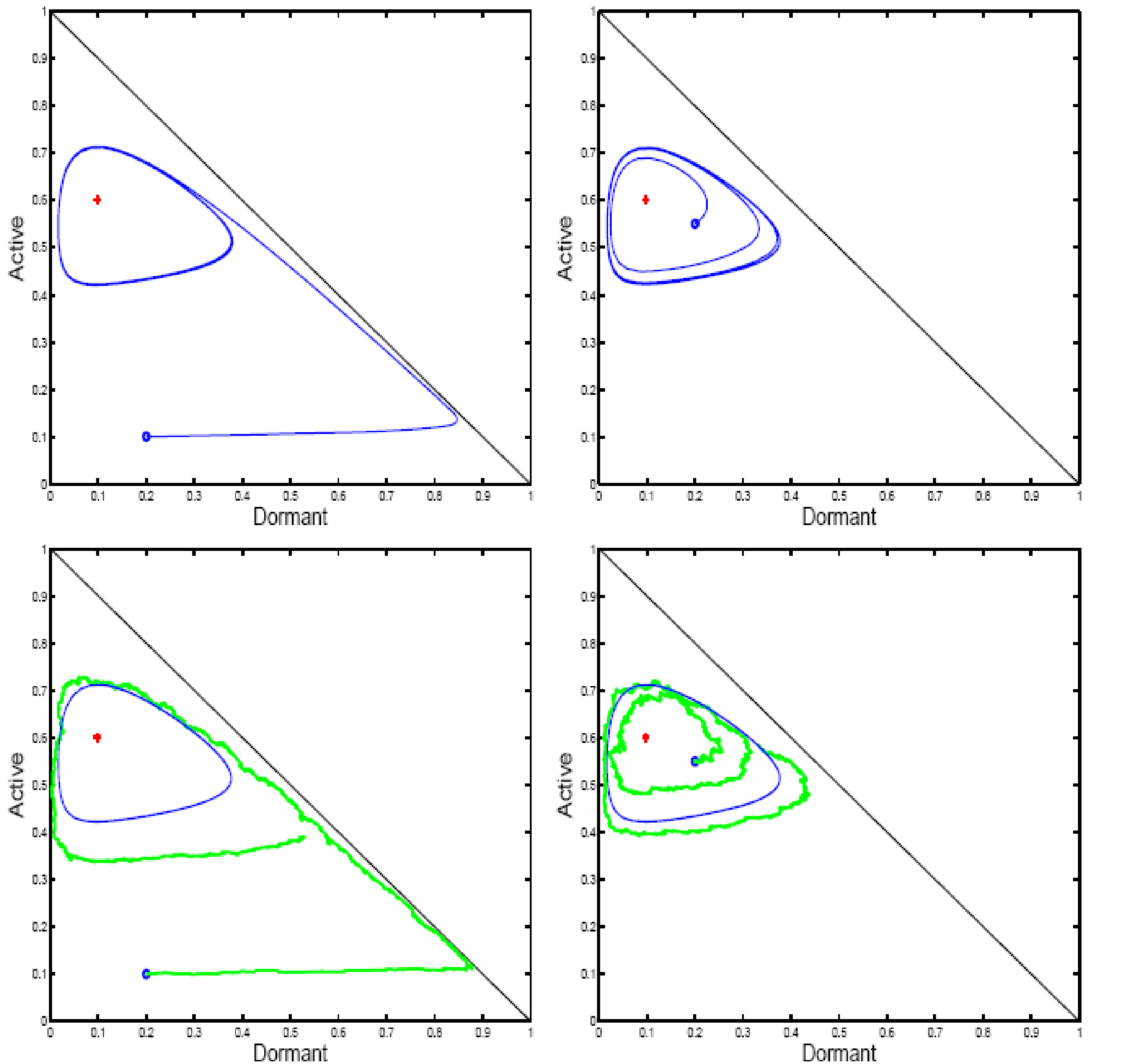
- Theorem [e.g. Benaim et al 2008] : The limit of stationary distribution of M^N is concentrated on this fixed point
- The decoupling assumption holds in stationary regime

Result 2: Birkhoff Center

- Here:
  Birkhoff center = limit cycle ⋃ fixed point

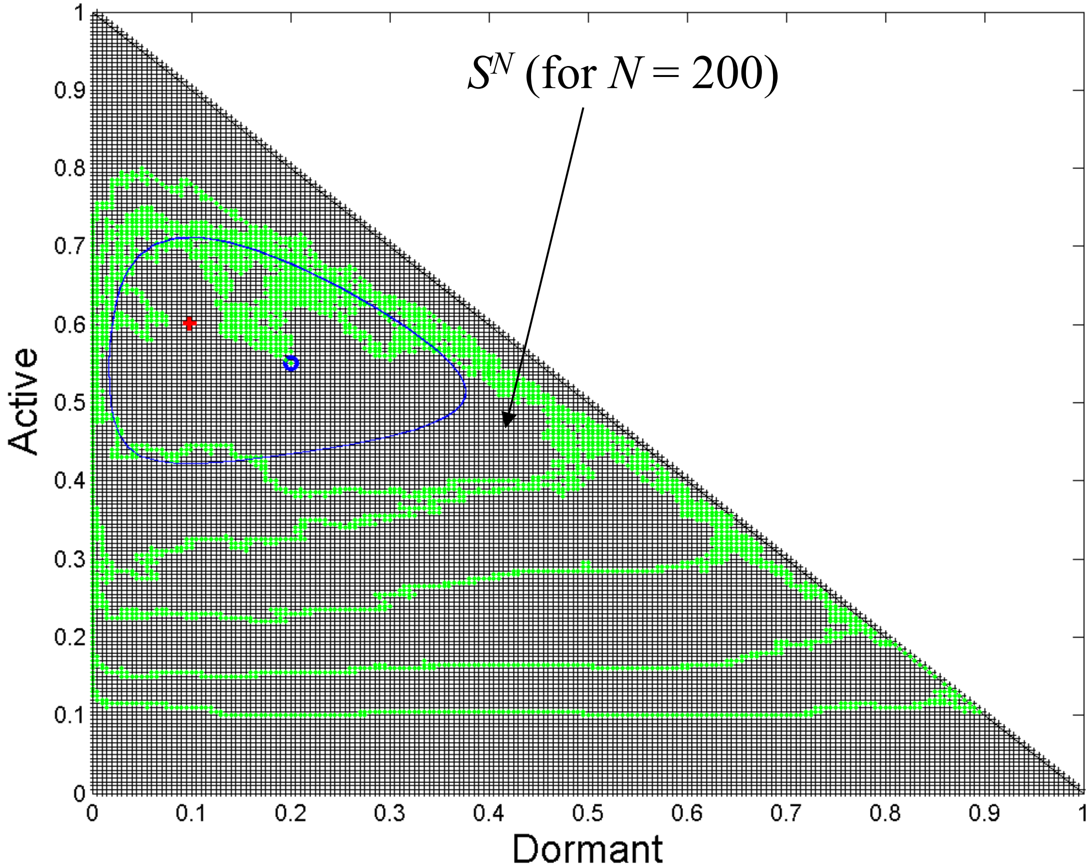
- Theorem in [Benaim] says that the stochastic system for large N is close to the Birkhoff center,

  i.e. the stationary regime of ODE is a good approximation of the stationary regime of stochastic system

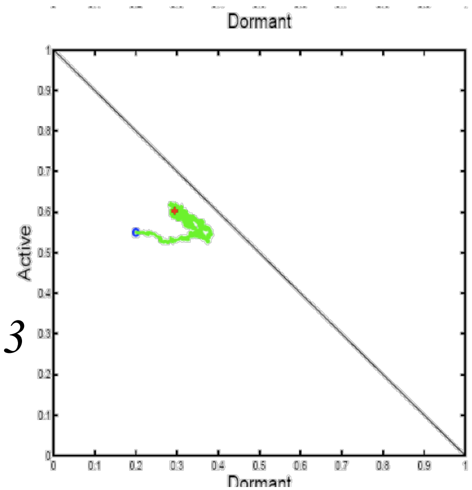


Stationary Behaviour of Mean Field Limit is not predicted by Structure of Markov Chain

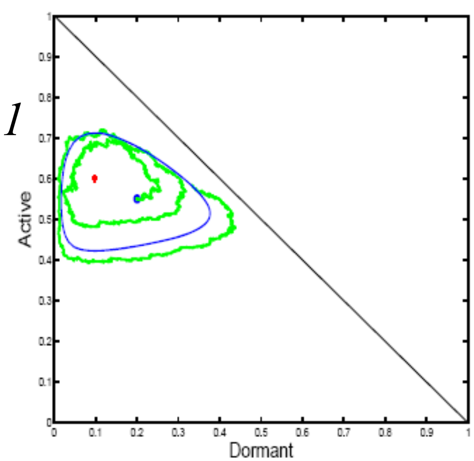
- $M^N(t)$ is a Markov chain on S^N={(a, b, c) ≥ 0, a + b + c =1, a, b, c multiples of 1/N}
- $M^N(t)$ is ergodic and aperiodic
- Depending on parameter, there is or is not a limit cycle for $m(t)$



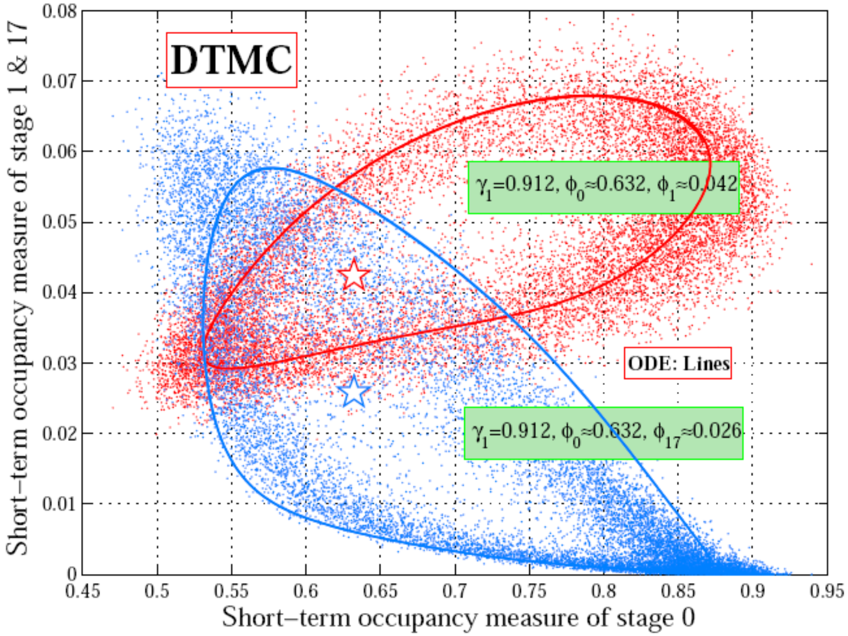
$h = 0.3$



$h = 0.1$



Example: 802.11 with Heterogeneous Nodes



- [Cho et al, 2010]

  Two classes of nodes with heterogeneous parameters (restransmission probability)

  Fixed point equation has a unique solution, but this is not the stationary proba

  There is a limit cycle

Result 3: In the Reversible Case, the Fixed Point Method Always Works

- **Definition** Markov Process X(t) on enumerable state *E* space, with transition rates *q(i,j)* is reversible iff

  1. it is ergodic 2. *p(i) q(i,j) = p(j) q(j,i)* for some *p*

Theorem 1.2 ([Le Boudec(2010)]) *Assume some process $Y^N(t)$ converges at any fixed t to some deterministic system $y(t)$ at any finite time. Assume the processes Y^N are reversible under some probabilities Π^N. Let $\Pi \in \mathcal{P}(E)$ be a limit point of the sequence Π^N. Π is concentrated on the set of stationary points S of the fluid limit $y(t)$*

- Stationary points = fixed points
- If process with finite *N* is reversible, the stationary behaviour is determined only by fixed points.

A Correct Method

- 1. Write dynamical system equations *in transient regime*

- 2. Study the *stationary regime of* dynamical system
  - **if** converges to unique stationary point m^* **then** make fixed point assumption
  - **else** objects are coupled in stationary regime by mean field limit $m(t)$

- Hard to predict outcome of 2 (except for reversible case)

Conclusion

- Mean field models are frequent in large scale systems
- Validity of approach is often simple by inspection
- Mean field is both
  - ODE for fluid limit
  - Fast simulation using decoupling assumption
- Decoupling assumption holds at finite horizon; may not hold in stationary regime.
- Stationary regime is more than stationary points, in general

  (except for reversible case)

Thank You …

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